Shift-invariance for FK-DLR states of a 2D quantum bose-gas

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Abstract

This paper continues the work [14] and focuses on infinite-volume bosonic states for a quantum system (a quantum gas) in a plane $\mathbb{R}^2$. We work under similar assumptions upon the form of local Hamiltonians and the type of the (pair) interaction potential as in [14]. The result of the paper is that any infinite-volume FK-DLR functional corresponding to the Hamiltonians is shift-invariant, regardless of whether this functional is unique or not.

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1. Introduction: FK-DLR states of quantum systems

This work continues [14] (and earlier works [5], [6] and [8]). The reference to [14] are marked by the Roman number **I**: Eqn (1.1.19.**I**), Theorem 1.1.**I**, Section 2.3.**I** and so on. In this paper we provide a justification of the notion of an FK-DLR (Feynman–Kac–Dobrushin–Lanford–Ruelle) state of a quantum system in an infinite volume (more generally, an FK-DLR functional of the quasi-local C*-algebra). The result of the present paper is that, for a quantum Bose-gas on a plane $\mathbb{R}^2$, any FK-DLR state is shift-invariant. This line of results takes its origin in [1], [9]; we want to stress that a particular impact upon the present work was made by Refs [10]–[12] (more credit will be given in due course).

We follow the background used in Sects 1.**I** – 3.**I**, in a specific situation where the dimension $d = 2$. Accordingly, $\Lambda = \Lambda_L$ and Λ_0 stand for squares $[-L, L]^{\times 2} \subset \mathbb{R}^2$ and $[-L_0, L_0]^{\times 2} \subset \mathbb{R}^2$, or – more generally – $[-L_0+\mathtt{c}^1, L_0+\mathtt{c}^1] \times [-L_0+\mathtt{c}^2, L_0+\mathtt{c}^2]$, where $c = (\mathtt{c}^1, \mathtt{c}^2) \in \mathbb{R}^2$ and $\Lambda \supset \Lambda_0$. As in [14], we denote by z and β the standard thermodynamical variables of the fugacity and the inverse temperature. The notions of a quantum n-particle Hamiltonian $H_{n,\Lambda}$ and the Gibbs state $\varphi_{z,\beta,\Lambda}$ in Λ are introduced as in Sects 1.1.**I** (see Eqns (1.1.1.**I**) – (1.1.25.**I**)). We also follow the conditions upon the two-body potential $V : [\mathtt{r}, \infty) \to \mathbb{R}$ imposed in [14]. (Here $\mathtt{r} \in (0, \infty)$ is the hard-core diameter, and we formally set $V(r) = +\infty$ for $0 \le r < \mathtt{r}$, conforming with the hard-core condition.) Moreover, we use the corresponding notation: cf. Eqns (1.1.3.**I**)–(1.1.5.**I**), (1.1.19.**I**) and (1.2.9.**I**). For the reader's convenience, we reproduce these conditions (and assume that they are valid throughout the paper):

$$V(r) = 0 \ \text{ for } \ r \ge \mathtt{R} \ \text{ where } \mathtt{R} \in (\mathtt{r}, \infty), \tag{1.1}$$

$$-\overline{V} = \min \left[V(r) : \ \mathtt{r} \le r \le \mathtt{R} \right], \tag{1.2}$$

with $\overline{V} = 0$ for $V \ge 0$,

$$\overline{V}^{(1)} = \max \left[|V'(r)| : \mathtt{r} \le r \le \mathtt{R} \right], \ \ \overline{V}^{(2)} = \max \left[|V''(r)| : \mathtt{r} \le r \le \mathtt{R} \right], \tag{1.3}$$

and

$$\overline{\rho} := z \exp\left(4\beta \overline{V} \mathtt{R}^2/\mathtt{r}^2\right) < 1. \tag{1.4}$$

The result from [14] implies that for $z>0$ and $\beta>0$ satisfying the bound (1.4), the family of Gibbs states $\{\varphi_{z,\beta,\Lambda}\}$ is compact and has limiting points as $\Lambda\nearrow\mathbb{R}^2$. Moreover, the family of Gibbs states $\{\varphi_{z,\beta,\Lambda|\mathbf{x}(\Lambda^c)}\}$ is compact where $\varphi_{z,\beta,\Lambda|\mathbf{x}(\Lambda^c)}$ is the Gibbs state in an external potential field generated by a 'classical' configuration $\mathbf{x}(\Lambda^c)\subset\Lambda^c$ satisfying (1.1.20.**I**). see Theorem 1.1. The limiting points for families $\{\varphi_{z,\beta,\Lambda}\}$ and $\{\varphi_{z,\beta,\Lambda|\mathbf{x}(\Lambda^c)}\}$ yield states of the quasi-local C*-algebra $\mathfrak{B}$; see (1.2.5.**I**). Such states describe possible 'thermodynamic phases' of the quantum Bose-gas in an infinite volume. A theory proposed in [14] goes a step further: we establish that any such limit-point state φ has a particular structure where the operators R^{Λ_0} yielding the (limiting) density matrices are constructed via an FK representation.

More precisely, the integral kernels $F^{\Lambda_0}(\mathbf{x}_0,\mathbf{y}_0)$ determining the density matrices R^{Λ_0} are written as integrals over spaces of so-called path and loop configurations; cf. Sect 2.**I**. An important rôle in these formulas is played by a probability measure (or probability measures) μ on $\mathcal{W}^*(\mathbb{R}^2)$, the space of loop configurations (LCs) in the plane $\mathbb{R}^2$. In a natural sense, the correspondence between a functional and a measure is one-to-one. Such a measure μ was called an FK-DLR probability measure (PM) and emerged as a limiting point for the family of similar measures in finite volumes Λ as $\Lambda\nearrow\mathbb{R}^2$. The set of FK-DLR PMs is denoted by $\mathfrak{K}=\mathfrak{K}(z,\beta)$; in a probabilistic terminology these measures are examples of random marked point processes (RMPPs) with marks represented by loops. Accordingly, a class of states $\mathfrak{F}_+=\mathfrak{F}_+(z,\beta)$ was introduced, called FK-DLR states, together with its enlargement, $\mathfrak{F}=\mathfrak{F}(z,\beta)\supset\mathfrak{F}_+$, giving a class of FK-DLR functionals on $\mathfrak{B}$. See Definitions 2.4.**I**–2.7.**I**. We stated a result, Theorem 1.2.**I**, and its generalization, Theorem 2.2.**I**, claiming that any functional from class $\mathfrak{F}$ is shift-invariant. For reader's convenience, we repeat here the statements of the latter.

The results of this paper are summarised in the following two theorems.

*The Fock spaces $\mathcal{H}(\Lambda_0)$ and $\mathcal{H}(\mathtt{S}(s)\Lambda_0)$ (cf. (1.1.12.**I**)) are related through a pair of mutually inverse shift isomorphisms of Fock spaces*

$$\mathtt{U}^{\Lambda_0}(s):\ \mathcal{H}(\Lambda_0)\to\mathcal{H}(\mathtt{S}(s)\Lambda_0)\ \textit{and}\ \mathtt{U}^{\Lambda_0}(-s):\ \mathcal{H}(\mathtt{S}(s)\Lambda_0)\to\mathcal{H}(\Lambda_0).$$

With the shift isometry $\mathtt{S}(\mathtt{s}):\mathbb{R}^2\to\mathbb{R}^2$:

$$\mathtt{S}(s)y=y+s,\ \ y\in\mathbb{R}^2,$$

and for the image $\mathtt{S}(s)\Lambda_0$ *of* Λ_0*:*

$$\mathtt{S}(s)\Lambda_0 = [\mathtt{b}^1 + \mathtt{s}^1 - L^0, \mathtt{b}^1 + \mathtt{s}^1 + L^0] \\ \times [\mathtt{b}^2 + \mathtt{s}^2 - L^0, \mathtt{b}^2 + \mathtt{s}^2 + L^0] .$$

The isomorphisms $\mathtt{U}^{\Lambda_0}(s)$ *and* $\mathtt{U}^{\Lambda_0}(-s)$ *are given by*

$$\begin{matrix} \left(\mathtt{U}^{\Lambda_0}(s)\phi_n\right)(\underline{x}_1^n) = \phi_n(\mathtt{S}(-s)\underline{x}_1^n), \\ \left(\mathtt{U}^{\Lambda_0}(-s)\phi_n\right)(\underline{x}_1^n) = \phi_n(\mathtt{S}(s)\underline{x}_1^n), \end{matrix} \quad \underline{x}_1^n \in (\Lambda_0)^n ,$$

where $\phi_n \in \mathrm{L}_2^{\mathrm{sym}}((\Lambda_0)^n)$, $n = 0, 1, \ldots$. Cf. Eqns (1.2.6.**I**)–(1.2.8.**I**).

Theorem 1.1. (cf. Theorem 2.2.**I**) *Assuming conditions* (1.1)–(1.4)*, let* μ *be a probability measure from* $\mathfrak{K}(z, \beta)$*. Then the corresponding FK-DLR functional* $\varphi_\mu \in \mathfrak{F}(z, \beta)$ *is shift-invariant: for any square* $\Lambda_0 \subset \mathbb{R}^2$*, vector* $s \in \mathbb{R}^2$ *and operator* $A \in \mathfrak{B}(\Lambda_0)$*,*

$$\varphi_\mu(U^{\mathtt{S}(s)\Lambda_0}(-s)AU^{\Lambda_0}) = \varphi_\mu(A).$$

In terms of the corresponding infinite-volume reduced density matrices R^{Λ_0}:

$$R^{\mathtt{S}(s)\Lambda_0} = \mathtt{U}^{\Lambda_0}(s)R^{\Lambda_0}\mathtt{U}^{\mathtt{S}(s)\Lambda_0}(-s).$$

In view of formulas (2.3.5.**I**)–(2.3.7.**I**) relating an FK-DLR functional φ to an FK-DLR PM μ, it suffices to verify

Theorem 1.2. *Any FK-DLR PM* μ *is translation invariant: for any* $s = (\mathtt{s}^1, \mathtt{s}^2) \in \mathbb{R}^2$*, square* $\Lambda_0 = [-L_0, L_0]^{\times 2}$ *and event* $\mathcal{D} \in \mathcal{W}^*(\mathbb{R}^2)$ *localised in* Λ_0 *(i.e., belonging to a sigma-algebra* $\mathfrak{W}(\Lambda_0)$*; cf Definition* 2.4.**I***),*

$$\mu(\mathtt{S}(s)\mathcal{D}) = \mu(\mathcal{D}).$$

Here $\mathtt{S}(s)\mathcal{D}$ *stands for the shifted event localised in the shifted square* $\mathtt{S}(s)\Lambda_0 = [-L_0 + \mathtt{s}^1, \mathtt{s}^1 + L_0] \times [-L_0 + \mathtt{s}^2, \mathtt{s}^2 + L_0]$.

2. Proof of Theorem 1.2: a tuned-shift argument

In what follows we use the terminology and the system of notation from Sect 1.**I** and 2.**I**. The proof of Theorem 1.2 is based on a modification of

an argument developed in [10]–[12]. We want to stress that the paper [12] treating some classes of (Gibbsian) RMPPs does not cover our situation because a number of the assumptions used in [12] are (unfortunately) not fulfilled here. Specifically, the condition (2.2) from [12] does not hold in our situation, as well as conditions specifying what is called a bpsi-function on p. 704 of [12].[1] The aforementioned modification demands that we use (and inspect) the construction from [11] for classical configurations (CCs) arising as $\mathtt{t}$-sections of LCs at a given time point $\mathtt{t}\in[0,\beta]$.

Because the argument in the proof does not depend on the direction of the vector s, we will assume that $s=(\mathtt{s},0)$ lies along the horizontal axis. Also, due to the group property, we can assume that $\mathtt{s}\in(0,1/2)$. By using constructions developed in [2] and [11]–[12], the assertion of Theorem 1.2 can be deduced from

Theorem 2.1. *Let μ be an FK-DLR PM, Λ_0 be a square $[-L_0,L_0]^{\times 2}$ and an event $\mathcal{D}\subset\mathcal{W}^*_{\mathtt{r}}(\mathbb{R}^2)$ be given, localized in Λ_0: $\mathcal{D}\in\mathfrak{W}(\Lambda_0)$. Then*

$$\mu(\mathtt{S}(s)\mathcal{D})+\mu(\mathtt{S}(-s)\mathcal{D})-2\mu(\mathcal{D})\geq 0. \tag{2.1}$$

For the proof of Theorem 2.1 we employ a strategy essentially mimicking the one from [10]–[12], particularly [11]. Consequently, we will follow the scheme from [11] rather closely, although, as was said earlier, we introduce considerable alterations. *We introduce the functionals K and L for path and loop configurations :*

$$K(\overline{\Omega}^*)=\sum_{\overline{\omega}^*\in\overline{\Omega}^*}k(\overline{\omega}^*),\quad K(\Omega^*)=\sum_{\omega^*\in\Omega^*}k(\omega^*),\quad L(\Omega^*)=\prod_{\omega^*\in\Omega^*}k(\omega^*).$$

For a given (large) L we introduce the square

$$\Lambda=[-L,L]\times[-L,L]\supset\Lambda_0 \tag{2.2}$$

and write the terms $\mu(\mathtt{S}(\pm s)\mathcal{D})$ and $\mu(\mathcal{D})$ as integrals of conditional expec-

[1] In short, the paper [12] employs an approach based on sup-norm conditions whereas the situation under consideration in this paper requires the use of integral-type norms. A crucial fact is that a Jacobian emerging in the course of the construction has the form (3.23) suitable for our purposes.

tations relative to the sigma-algebra $\mathfrak{W}(\Lambda^{\rm c})$:

$$\begin{array}{l}\displaystyle\int_{\mathcal{W}^*_{\tt r}(\mathbb{R}^2)}\mu({\rm d}\mathbf{\Omega}^*_{\Lambda^{\rm c}})\mathbf{1}\Big(\mathbf{\Omega}^*_{\Lambda^{\rm c}}\in\mathcal{W}_{\tt r}\left(\Lambda^{\rm c}\right)\Big)\\ \qquad\displaystyle\times\int_{\mathcal{W}^*_{\tt r}(\Lambda)}{\rm d}\Omega^*_\Lambda\mathbf{1}\Big(\Omega^*_\Lambda\in{\tt S}(\pm s)\mathcal{D}\Big)\frac{z^{K(\Omega^*_\Lambda)}}{L(\Omega^*_\Lambda)}\exp\,\big[-h(\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^{\rm c}})\,\big]\end{array}\tag{2.3}$$

(the case of $\mu(\mathcal{D})$ is recovered at $s=0$, with ${\tt S}(0)={\rm Id}$.)

Furthermore, again as in [10]–[12], we employ maps ${\tt T}^\pm_L={\tt T}^\pm_{L,L_0}(s):\mathcal{W}^*(\mathbb{R}^2)\to\mathcal{W}^*_{\tt r}(\mathbb{R}^2)$.[2] These are applied to the concatenated loop configuration (LC) $\Omega^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\rm c}}$ in the expressions from Eqn (4.3), in the the corresponding case of shift ${\tt S}(\pm s)$. Important properties of maps ${\tt T}^\pm_L$ are:

(i) The maps $(\Omega^*_\Lambda,\mathbf{\Omega}^*_{\Lambda^{\rm c}})\mapsto{\tt T}^\pm_L(\Omega^*_\Lambda,\mathbf{\Omega}^*_{\Lambda^{\rm c}})$ are one-to-one, and a number of 'nice' properties hold true when the LC $\Omega^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\rm c}}$ lies in a 'good' set $\mathcal{G}_L\subset\mathcal{W}^*_{\tt r}(\mathbb{R}^2)$. (Viz., for $\Omega^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\rm c}}\in\mathcal{G}_L$ the loops from $\Omega^*_\Lambda\cap\mathcal{W}^*_{\tt r}(\Lambda_0)$ will not interact with loops from $\mathbf{\Omega}^*_{\Lambda^{\rm c}}$.) The set $\mathcal{G}_L$ carries asymptotically a full measure as $L\to\infty$. See below.

(ii) For a 'good' LC $\mathbf{\Omega}^*=\mathbf{\Omega}^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\rm c}}\in\mathcal{G}_L$ over $\mathbb{R}^2$, the 'external' part $\mathbf{\Omega}^*_{\Lambda^{\rm c}}$ is preserved under ${\tt T}^\pm_n$. In other words, the maps are non-trivial only on the part $\mathbf{\Omega}^*_\Lambda$ (although the way $\mathbf{\Omega}^*_\Lambda$ is transformed depends upon $\mathbf{\Omega}^*_{\Lambda^{\rm c}}$ (and on $\mathbf{\Omega}^*_\Lambda$, of course)). For that reason we will often address ${\tt T}^\pm_L$ as a 'tuned' shift $\mathbf{\Omega}^*\mapsto\widetilde{\mathbf{\Omega}}^*=({\tt T}^\pm_L\mathbf{\Omega}^*_\Lambda)\vee\mathbf{\Omega}^*_{\Lambda^{\rm c}}$ or, dealing with a pair $(\Omega^*_\Lambda,\mathbf{\Omega}^*_{\Lambda^{\rm c}})\in\mathcal{W}^*(\Lambda,\Lambda^{\rm c})$,

$$\Omega^*_\Lambda\mapsto\widetilde{\Omega}^*_\Lambda={\tt T}^\pm_L\Omega^*_\Lambda\in\mathcal{W}^*_{\tt r}(\Lambda).\tag{2.4}$$

With this agreement:

(iii) The transformation (2.4) preserves the cardinality: $\sharp\,\Omega^*_\Lambda=\sharp\,\widetilde{\Omega}^*_\Lambda$ and transforms a loop $\omega^*\in\Omega^*_\Lambda$ as $\omega^*\mapsto\widetilde{\omega}^*$ where $k(\widetilde{\omega}^*)=k(\omega^*)$. Consequently, functionals K and L are preserved: $K(\widetilde{\Omega}^*)=K(\Omega^*)$ and $L(\widetilde{\Omega}^*)=L(\Omega^*)$. Next, for all ${\tt t}\in[0,k(\omega^*)\beta]$, point $\widetilde{\omega}^*({\tt t})\in\mathbb{R}^2$ is obtained as a 'tuned shift'

$$\widetilde{\omega}^*({\tt t})=\omega^*({\tt t})\pm s{\tt R}^\pm_L\Big[\omega^*;{\tt t};\{\Omega^*_\Lambda\}({\tt t})\cup\{\mathbf{\Omega}^*_{\Lambda^{\rm c}}\}({\tt t})\Big];\tag{2.5}$$

see below. We stress that the argument of function ${\tt R}^\pm_L$ consists of a loop $\omega^*\in\mathcal{W}^*_{\tt r}$, a time point ${\tt t}\in[0,k(\omega^*)\beta]$ and the ${\tt t}$-section $\{\Omega^*_\Lambda\}({\tt t})\cup\{\mathbf{\Omega}^*_{\Lambda^{\rm c}}\}({\tt t})=$

[2]The symbol used in [10]–[12] is $\mathfrak{T}$ instead of ${\tt T}$. The idea of using maps ${\tt T}^\pm_L$ goes back to [1] and [9].

$\{\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c}\}(\mathtt{t}) \in \mathcal{C}_\mathtt{r}(\mathbb{R}^2)$ of an LC $\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c}$. Here, as in [14], $\mathcal{C}(\mathbb{R}^2)$ stands for the collection of finite or countable (unordered) subsets $\mathbf{x} \subset \mathbb{R}^2$ (including the empty set) and $\mathcal{C}_\mathtt{r}(\mathbb{R}^2) \subset \mathcal{C}(\mathbb{R}^2)$ for the collection of subsets $\mathbf{x}$ with

$$\min \Big[|x-x'|:\ x,x' \in \mathbf{x},\ x \neq x'\Big] \geq \mathtt{r}.$$

(iv) For simplicity, let us omit henceforth the symbols $\pm$ whenever possible. The value $\mathtt{w}_L\Big[\omega^*;\mathtt{t};\{\Omega^*_\Lambda\}(\mathtt{t}) \cup \{\mathbf{\Omega}^*_{\Lambda^c}\}(\mathtt{t})\Big]$ is a non-negative number. Moreover, when $\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c} \in \mathcal{G}_L$ then for $\omega^* \in \Omega^*_\Lambda \cap \mathcal{W}^*_\mathtt{r}(\Lambda_0)$ and $0 \leq \mathtt{t} \leq k(\omega^*)\beta$,

$$\mathtt{w}_L\Big[\omega^*;\mathtt{t};\{\Omega^*_\Lambda\}(\mathtt{t}) \cup \{\mathbf{\Omega}^*_{\Lambda^c}\}(\mathtt{t})\Big] \equiv 1,\ \ 0 \leq \mathtt{t} \leq k(\omega^*)\beta.$$

Consequently, in accordance with (2.5), for $\omega^* \in \mathcal{W}^*_\mathtt{r}(x)$ with $x \in \Lambda_0$ and $\mathtt{t} \in [0, k(\omega^*)\beta]$ the point $\widetilde{\omega}^*(\mathtt{t}) = \omega^*(\mathtt{t}) + s$. Therefore, the loops ω^* from $\Omega^*_0 = \Omega^*_\Lambda \cap \mathcal{W}^*_\mathtt{r}(\Lambda_0)$ are shifted intact by the amount s under the map (2.4). Consequently, the integral energy $h(\Omega^*_0)$ is not changed under tuned shifts.

(v) The set $\mathtt{S}(s)(\mathcal{D} \cap \mathcal{G}_L)$ will have a μ-measure close to that of $\mathtt{S}(s)\mathcal{D}$; moreover, the probability $\mu(\mathtt{S}(s)(\mathcal{D} \cap \mathcal{G}_L))$ will be written in the form

$$\begin{aligned}\mu(\mathtt{S}(\pm s)(\mathcal{D} \cap \mathcal{G}_L)) &= \int_{\mathcal{W}^*_\mathtt{r}(\mathbb{R}^2)} \mu(\mathrm{d}\mathbf{\Omega}^*_{\Lambda^c})\mathbf{1}\Big(\mathbf{\Omega}^*_{\Lambda^c} \in \mathcal{W}_\mathtt{r}(\Lambda^c)\Big)\\ &\times \int_{\mathcal{W}^*_\mathtt{r}(\Lambda)} \mathrm{d}\Omega^*_\Lambda\ \mathbf{1}\Big(\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c} \in \mathcal{G}_L \cap \mathcal{D}\Big)\ \frac{z^{K(\Omega^*_\Lambda)}}{L(\Omega^*_\Lambda)}\\ &\times J^\pm_L(\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c}) \exp\ \Big[-h(\mathtt{T}^\pm_L(s)\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^c})\Big]\end{aligned} \tag{2.6}$$

where function $J^\pm_L = J^\pm_{L,s}$ gives the Jacobian of transformation $\mathtt{T}^\pm_L(s)$. By virtue of properties above (cf. particularly (i) and (iv)), the impact of tT_L upon the energy $h(tT_L\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^c})$ will be felt through the LC $\Omega^*_{\Lambda\setminus\Lambda_0} = \Omega^*_\Lambda \cap \mathcal{W}^*_\mathtt{r}(\Lambda \setminus \Lambda^0)$ only. (More precisely, through a LC $\Omega^*_{\Lambda\setminus\Lambda_{R(L)}}$ where $\Lambda_{R(L)} = [-R(L), R(L)]^{\times 2}$ and $R(L) \nearrow \infty$ with L. See Eqn (3.2) below.) Essentially, the same remains true about the Jacobian $J_L(\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c})$.

(vi) In fact, a detailed analysis shows that second-order incremental expressions

$$\Big[J^+_L(\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c})J^-_L(\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c})\Big]^{1/2} \tag{2.7}$$

and

$$\exp\ \Big[h(\mathtt{T}_L^+(s)\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^c}) + h(\mathtt{T}_L^-(s)\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^c}) - 2h(\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^c})\Big] \tag{2.8}$$

are close to 1. It turns out that this fact suffices for the assertion of Theorem 2.1.

Formally, Theorem 2.1 is derived from

Theorem 2.2. *For any $\delta > 0$ there exists $L_0^* = L_0^*(\delta) > 0$ such that for $L \geq L_0^*$*

(A)
$$\begin{aligned}\mu(\mathcal{G}_L) = \int_{\mathcal{W}^*_{\mathtt{r}}(\mathbb{R}^2)} &\mu(\mathrm{d}\Omega^*_{\Lambda^c})\ \mathbf{1}\Big(\Omega^*_{\Lambda^c} \in \mathcal{W}^*_{\mathtt{r}}(\Lambda^c)\Big)\\ \times \int_{\mathcal{W}^*_{\mathtt{r}}(\Lambda)} &\mathrm{d}\Omega^*_\Lambda\ \mathbf{1}\Big(\Omega^*_\Lambda \vee \Omega^*_{\Lambda^c} \in \mathcal{G}_L\Big)\\ &\times \frac{z^{K(\Omega^*_\Lambda)}}{L(\Omega^*_\Lambda)}\ \exp\ \big[-h(\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^c})\big] \geq 1-\delta.\end{aligned} \tag{2.9}$$

(B) *The probabilities $\mu(\mathtt{S}(\pm s)(\mathcal{D} \cap \mathcal{G}_L))$ are represented in the form (2.6) with the following properties: $\forall\ \Omega^*_\Lambda \in \mathcal{W}^*_{\mathtt{r}}(\Lambda)$, $\mathbf{\Omega}^*_{\Lambda^c} \in \mathcal{W}^*_{\mathtt{r}}(\Lambda^c)$ with $\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c} \in \mathcal{G}_L$;*

(Ca) $\Big[J_L^+(\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c}) J_L^-(\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c})\Big]^{1/2} \geq 1-\delta$;

(Cb) $h(\mathtt{T}_L^+(s)\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^c}) + h(\mathtt{T}_L^-(s)\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^c}) - 2h(\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^c}) \leq \delta$.

The proof of Theorem 2.2 is carried on in the next sections.

Remark. It is the pair of inequalities (Ca), (Cb) (together with the definition of the 'good' set $\mathcal{G}_L$) where one crucially uses the fact that the physical dimension of the system equals 2.

We now show how to deduce the statement of Theorem 2.1 from that of Theorem 2.2. Owing to Theorem 2.2 (A), (B), we can write:

$$\begin{aligned}&\text{the LHS of (2.1)} + 3\delta\\ &\quad\geq \mu(\mathtt{S}(s)(\mathcal{D}\cap\mathcal{G}_L)) + \mu(\mathtt{S}(-s)(\mathcal{D}\cap\mathcal{G}_L)) - 2\mu(\mathcal{D}\cap\mathcal{G}_L)\\ &\quad= \int_{\mathcal{W}^*_{\mathtt{r}}(\mathbb{R}^2)} \mu(\mathrm{d}\mathbf{\Omega}^*_{\Lambda^c})\mathbf{1}\Big(\mathbf{\Omega}^*_{\Lambda^c} \in W_{\mathtt{r}}(\Lambda^c)\Big)\\ &\qquad\times \int_{\mathcal{W}^*_{\mathtt{r}}(\Lambda)} \mathrm{d}\Omega^*_\Lambda\ \mathbf{1}\Big(\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c} \in \mathcal{G}_L \cap \mathcal{D}\Big)\ \frac{z^{K(\Omega^*_\Lambda)}}{L(\Omega^*_\Lambda)}\\ &\quad\times\Big\{J_L^+(\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c})\exp\ \big[-h(\mathtt{T}_L^+\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^c})\big]\\ &+J_L^-(\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c})\exp\ \big[-h(\mathtt{T}_L^-\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^c})\big] - 2\exp\ \big[-h(\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^c})\big]\Big\}.\end{aligned} \tag{2.10}$$

Next, by the AM/GM inequality, the RHS of (2.10) is no less than

$$
\begin{aligned}
&2\int_{\mathcal{W}^*_{\mathtt{r}}(\mathbb{R}^2)}\mu(\mathrm{d}\boldsymbol{\Omega}^*_{\Lambda^{\mathrm{c}}})\mathbf{1}\Big(\boldsymbol{\Omega}^*_{\Lambda^{\mathrm{c}}}\in W_{\mathtt{r}}\,(\Lambda^{\mathrm{c}})\Big)\int_{\mathcal{W}^*_{\mathtt{r}}(\Lambda)}\mathrm{d}\Omega^*_\Lambda\;\mathbf{1}\Big(\Omega^*_\Lambda\vee\boldsymbol{\Omega}^*_{\Lambda^{\mathrm{c}}}\in\mathcal{G}_L\cap\mathcal{D}\Big)\\
&\qquad\times\frac{z^{K(\Omega^*_\Lambda)}}{L(\Omega^*_\Lambda)}\ \Big(\Big[J^+_L(\Omega^*_\Lambda\vee\boldsymbol{\Omega}^*_{\Lambda^{\mathrm{c}}})J^-_L(\Omega^*_\Lambda\vee\boldsymbol{\Omega}^*_{\Lambda^{\mathrm{c}}})\Big]^{1/2}\\
&\times\exp\Big\{-\big[h(\mathtt{T}^+_L\Omega^*_\Lambda|\boldsymbol{\Omega}^*_{\Lambda^{\mathrm{c}}})+h(\mathtt{T}^-_L\Omega^*_\Lambda|\boldsymbol{\Omega}^*_{\Lambda^{\mathrm{c}}})\big]/2\Big\}-\exp\ \big[-h(\Omega^*_\Lambda|\boldsymbol{\Omega}^*_{\Lambda^{\mathrm{c}}})\big]\Big).
\end{aligned}
\tag{2.11}
$$

Now, by virtue of Theorem 2.2 (A)–(C), the RHS of (2.11) is greater than or equal to

$$
\begin{aligned}
&2[(1-\delta)e^{-\delta/2}-1]\int_{\mathcal{W}^*_{\mathtt{r}}(\mathbb{R}^2)}\mu(\mathrm{d}\boldsymbol{\Omega}^*_{\Lambda^{\mathrm{c}}})\mathbf{1}\Big(\boldsymbol{\Omega}^*_{\Lambda^{\mathrm{c}}}\in W_{\mathtt{r}}\,(\Lambda^{\mathrm{c}})\Big)\\
&\qquad\times\int_{\mathcal{W}^*_{\mathtt{r}}(\Lambda)}\mathrm{d}\Omega^*_\Lambda\;\mathbf{1}\Big(\Omega^*_\Lambda\vee\boldsymbol{\Omega}^*_{\Lambda^{\mathrm{c}}}\in\mathcal{G}_L\cap\mathcal{D}\Big)\\
&\qquad\qquad\times\frac{z^{K(\Omega^*_\Lambda)}}{L(\Omega^*_\Lambda)}\exp\ \big[-h(\Omega^*_\Lambda|\boldsymbol{\Omega}^*_{\Lambda^{\mathrm{c}}})\big]\\
&\quad=2[(1-\delta)e^{-\delta/2}-1]\mu(\mathcal{G}_L\cap\mathcal{D})\\
&\qquad\geq 2[(1-\delta)e^{-\delta/2}-1](1-\delta).
\end{aligned}
\tag{2.12}
$$

Since δ can be made arbitrarily small, we obtain the inequality (2.2).

3. Definition of transformations $\mathtt{T}^{\pm}_L$

As was said earlier, the maps $\boldsymbol{\Omega}^*\mapsto\mathtt{T}^{\pm}_L\boldsymbol{\Omega}^*_\Lambda\vee\boldsymbol{\Omega}^*_{\Lambda^{\mathrm{c}}}$ are determined by transforming the $\mathtt{t}$-sections $\{\mathtt{T}^{\pm}_L\boldsymbol{\Omega}^*_\Lambda\}(\mathtt{t})$ of the LC $\boldsymbol{\Omega}^*_\Lambda$, for each $\mathtt{t}\in[0,\beta]$. Denoting by $T^{\pm}_L=T^{\pm}_L(\pm s)$ the map acting on CCs from $\mathcal{C}_{\mathtt{r}}(\Lambda)$, we can write:

$$
\begin{aligned}
\{\mathtt{T}^{\pm}_L\boldsymbol{\Omega}^*_\Lambda\vee\boldsymbol{\Omega}^*_{\Lambda^{\mathrm{c}}}\}(\mathtt{t})\ &=\{\mathtt{T}^{\pm}_L\boldsymbol{\Omega}^*_\Lambda\}(\mathtt{t})\vee\boldsymbol{\Omega}^*_{\Lambda^{\mathrm{c}}}(\mathtt{t})\\
&=\big(T^{\pm}_L[\{\boldsymbol{\Omega}^*_\Lambda\}(\mathtt{t})]\big)\vee\boldsymbol{\Omega}^*_{\Lambda^{\mathrm{c}}}(\mathtt{t}).
\end{aligned}
\tag{3.1}
$$

Like before, we would like to stress that the way the $\mathtt{t}$-section $\{\boldsymbol{\Omega}^*_\Lambda\}(\mathtt{t})$ is transformed depends on $\{\boldsymbol{\Omega}^*_{\Lambda^{\mathrm{c}}}\}(\mathtt{t})$, although $\{\boldsymbol{\Omega}^*_{\Lambda^{\mathrm{c}}}\}(\mathtt{t})$ itself is not moving when $\boldsymbol{\Omega}^*\in\mathcal{G}_L$.

More precisely, set:

$$
R(L)=\big(\log\,\log\,\ L\big)^{3/4},\ \ \Lambda_{R(L)}=[-R(L),R(L)]^{\times 2},
\tag{3.2}
$$

and introduce yet another intermediate square

$$\overline{\Lambda}=[-\overline{L},\overline{L}]^{\times 2}\ \text{ where }\ \overline{L}=L-L^{3/4}. \tag{3.3}$$

We will assume that the quadruple of squares Λ_0, $\Lambda_{R(L)}$, $\overline{\Lambda}$ and Λ satisfies

$$\Lambda_0\subset\Lambda_{R(L)}\subset\overline{\Lambda}\subset\Lambda.$$

The transformed CC $T^{\pm}_L[\{\mathbf{\Omega}^*_\Lambda\}(\mathtt{t})]\in\mathcal{C}_{\mathtt{r}}(\Lambda)$ is formed by points $\widetilde{\omega}^{*\pm}_L(l\beta+\mathtt{t})$ obtained, as a result of shifts in the (positive) horizontal direction, from the points $\omega^*(l\beta+\mathtt{t})$ where $\mathtt{t}\in[0,\beta]$, $l=0,\dots,k(\omega^*)-1$ and $\omega^*\in\mathbf{\Omega}^*_\Lambda$:

$$\widetilde{\omega}^{*\pm}_L(l\beta+\mathtt{t})=\omega^*(l\beta+\mathtt{t})\pm\mathtt{p}_L(\omega^*(l\beta+\mathtt{t}))s. \tag{3.4}$$

Here the scalar value $\mathtt{p}_L(\omega^*(l\beta+\mathtt{t}))\geq 0$ depends on CCs $\{\mathbf{\Omega}^*_\Lambda\}(\mathtt{t})$ and $\{\mathbf{\Omega}^*_{\Lambda^c}\}(\mathtt{t})$ and are constructed recursively; cf. [11]. When $\omega^*(l\beta+\mathtt{t})\in\Lambda\setminus\overline{\Lambda}$, we have that

$$\mathtt{p}_L(\omega^*(l\beta+\mathtt{t}))=0\ \text{ and }\ \widetilde{\omega}^{*\pm}_L(l\beta+\mathtt{t})=\omega^*(l\beta+\mathtt{t}).$$

In other words, a loop $\omega^*\in\mathbf{\Omega}^*$ is affected only at points $\omega^*(\mathtt{t})$ lying in $\overline{\Lambda}$.

In the course of construction of values $\mathtt{p}_L(\omega^*(l\beta+\mathtt{t}))$, we employ the function $u\in[0,\infty)\mapsto\tau_L(u)$ determined as follows:

$$\tau_L(u)=\begin{cases}1, & 0\leq u\leq R(L),\\ 1-\dfrac{Q(u-R(L))}{Q(L-R(L))}, & R(L)\leq u\leq\overline{L},\\ 0, & u\geq\overline{L},\end{cases} \tag{3.5}$$

where, in turn,[3]

$$Q(u)=\int_0^u q(v)\mathrm{d}v,\ \text{ with }\ q(v)=\frac{1}{1\vee v|\log v|}. \tag{3.6}$$

The values $\mathtt{p}(\omega^*(l\beta+\mathtt{t}))=\mathtt{p}_L(\omega^*(l\beta+\mathtt{t}))$ are related to results of a series of minimizations, over points $\omega^*(l\beta+\mathtt{t})\in\{\mathbf{\Omega}^*\}(\mathtt{t})\cap\overline{\Lambda}$, of subsequently introduced functions $\widetilde{t}^{\,(j)}(\,\cdot\,;\mathtt{t})=\widetilde{t}^{\,(j)}_L(\,\cdot\,;\mathtt{t})$. Here j runs from 0 to $\sharp\,\left(\{\mathbf{\Omega}^*\}(\mathtt{t})\cap\overline{\Lambda}\right)$ and the functions are

$$\begin{gathered}y\in\mathbb{R}^2\mapsto\widetilde{t}^{\,(j)}(y;\mathtt{t})\in[0,1],\\ 0\leq j\leq\sum_{\omega^*\in\mathbf{\Omega}^*}\sum_{0\leq l<k(\omega^*)}\mathbf{1}(\omega^*(l\beta+\mathtt{t})\in\Lambda).\end{gathered} \tag{3.7}$$

[3] Function τ_L was introduced in [1] and [9] and has been repeatedly used in the literature.

The value $j = 0$ marks an initial function $t^{(0)}(\,\cdot\,;\mathtt{t})$, and the values $j \geq 1$ provide an ordering for points $\omega^*(l\beta+\mathtt{t})$ in the CC $\{\boldsymbol{\Omega}^*\}(\mathtt{t})\cap\overline{\Lambda}$. Let us stress that the functions $\widetilde{t}_L^{(j)}(\,\cdot\,;\mathtt{t})$ involve (generally speaking) the whole $\mathtt{t}$-section $\{\boldsymbol{\Omega}^*\}(\mathtt{t})$.

The initial function in the series, $\widetilde{t}_L^{(0)}(\,\cdot\,;\mathtt{t})$, does not depend on $\mathtt{t} \in [0,\beta]$ and is related to function $\tau = \tau_L$ from (3.5):

$$\widetilde{t}^{(0)}(y;\mathtt{t}) := \tau\left(|y|_{\rm m}\right). \tag{3.8}$$

Here and below, $|\cdot|_{\rm m}$ stands for the max-norm: $|y|_{\rm m} = \max\left[\left|\mathtt{y}^{(1)}\right|, \left|\mathtt{y}^{(2)}\right|\right]$, for $y = \left(\mathtt{y}^{(1)}, \mathtt{y}^{(2)}\right)$.

The definition of the next function, $\widetilde{t}_L^{(1)}(\,\cdot\,;\mathtt{t})$, involves a (multiple) minimum of auxiliary functions $m_{x,0}$, over the points $x = \omega^*(l\beta+\mathtt{t})$ from the CC $\{\boldsymbol{\Omega}^*\}(\mathtt{t})\cap\overline{\Lambda}^{\rm c}$:

$$\widetilde{t}^{(1)}(y;\mathtt{t}) = \widetilde{t}^{(0)}(y;\mathtt{t}) \wedge \widetilde{m}^{(0)}(y;\mathtt{t}) \tag{3.9}$$

where

$$\widetilde{m}^{(0)}(y;\mathtt{t}) = \widetilde{m}_L^{(0)}(y;\mathtt{t}) = \bigwedge_{\omega^*(l\beta+\mathtt{t})\in\left\{\boldsymbol{\Omega}^*\right\}(\mathtt{t})\cap\overline{\Lambda}^{\rm c}} m_{\omega^*(l\beta+\mathtt{t}),0}\,(y). \tag{3.10}$$

Here and below, following [9], [1], [11], the family of auxiliary functions $y \in \mathbb{R}^2 \mapsto m_{x,\mathtt{u}}(y)$ is used, with values in $[0,1)\cup\{+\infty\}$, where $x \in \mathbb{R}^2$, $\mathtt{u} \in [0,1)$. These functions are introduced as follows:

$$m_{x,\mathtt{u}}(y) := \begin{cases} \mathtt{u}, & \mathtt{h}_{x,\mathtt{u}}\mathtt{c}_f > 1/2, \\ \mathtt{u} + \mathtt{h}_{x,\mathtt{u}} f(x-y) & \\ \qquad +\infty\cdot\mathbf{1}(f(x-y)=1), & \mathtt{h}_{x,\mathtt{r}}\mathtt{c}_f \leq 1/2. \end{cases} \tag{3.11}$$

In turn, $f = f_\epsilon$ is a chosen C^1-function $\mathbb{R}^2 \to [0,1]$, with

$$f(v) = 0 \;\text{ when }\; |v| < a \;\text{ and }\; f(v) = 1 \;\text{ when }\; |v| > a + 2\epsilon,$$

and

$$\mathtt{c}_f = \max\left[|\nabla f(v)|,\ v \in \mathbb{R}^2\right]. \tag{3.12}$$

The value ϵ is selected for given z and β satisfying (1.4) and should be small enough, guaranteeing smallness of quantities introduced below. Finally,

$$\mathtt{h}_{x,\mathtt{u}} := |\tau(|x|_{\rm m} - \epsilon - a/2) - \mathtt{u}\,| \tag{3.13}$$

is another auxiliary parameter.

Pictorially speaking, the function $y \in \mathbb{R}^2 \mapsto \widetilde{m}^{(0)}(y;\mathtt{t})$ indicates by how much a particle (i.e., a circle of diameter $\mathtt{r}$) placed at the reference point y could be moved (under adopted arrangements) in presence of hard-core particles placed at points $\omega^*(l\beta+\mathtt{t}) \in \{\boldsymbol{\Omega}^*\}(\mathtt{t}) \cap \overline{\Lambda}^{\mathrm{c}}$. Consequently, $\widetilde{t}_L^{(1)}(y;\mathtt{t})$ indicates how much a movement by quantity $\widetilde{t}_L^{(0)}(y;\mathtt{t})$ should be reduced in presence of hard-core particles at $\omega^*(l\beta+\mathtt{t}) \in \{\boldsymbol{\Omega}^*\}(\mathtt{t}) \cap \overline{\Lambda}^{\mathrm{c}}$. We then look for the minimum of $\widetilde{t}_L^{(1)}(\,\cdot\,;\mathtt{t})$ over the CC $\{\boldsymbol{\Omega}^*\}(\mathtt{t}) \cap \overline{\Lambda}$ and set:

$$\begin{aligned} \mathtt{p}^1 = \mathtt{p}^1_L &= \min \left[\widetilde{t}_L^{(1)}(y;\mathtt{t}):\ y \in \{\boldsymbol{\Omega}^*\}(\mathtt{t}) \cap \overline{\Lambda}\right], \\ P^1 = P^1_L &= \arg\min \left[\widetilde{t}_L^{(1)}(y;\mathtt{t}):\ y \in \{\boldsymbol{\Omega}^*\}(\mathtt{t}) \cap \overline{\Lambda}\right]. \end{aligned} \tag{3.14}$$

If the minimum is attained at more than one point in $\{\boldsymbol{\Omega}^*\}(\mathtt{t}) \cap \overline{\Lambda}$, we list all these points: $P^1, \ldots, P^{\kappa_1}$ (in any order). The value $\mathtt{p}^1$ is assigned to each of those points as $\mathtt{p}(P^j)$:

$$\begin{aligned} \mathtt{p}(\omega^*(l\beta+\mathtt{t})) = \mathtt{p}^1,\ \ &\text{if}\ \ \omega^* \in \boldsymbol{\Omega}^*, 0 \le l < k(\omega^*), \\ &\omega^*(l\beta+\mathtt{t}) \in \overline{\Lambda}\ \ \text{and}\ \ \widetilde{t}^{(1)}(\omega^*(l\beta+\mathtt{t});\mathtt{t}) = \mathtt{p}^1. \end{aligned} \tag{3.15}$$

The value $\mathtt{p}^1$ and the position P^1 (or the positions $P^1, P^2, \ldots, P^{\kappa_1}$) are taken into account in the definition of the next function $y \in \mathbb{R}^2 \mapsto \widetilde{t}^{(2)}(y;\mathtt{t})$:

$$\begin{aligned} \widetilde{t}^{(2)}(y;\mathtt{t}) &= \widetilde{t}^{(1)}(y;\mathtt{t}) \wedge m_{P^1,\mathtt{p}^1\times\mathtt{s}}(y) \ldots \wedge m_{P^{\kappa_1},\mathtt{p}^1\times\mathtt{s}}(y) \\ &= \widetilde{t}^{(0)}(y;\mathtt{t}) \wedge \widetilde{m}^{(1)}(y;\mathtt{t}). \end{aligned} \tag{3.16}$$

Here $\widetilde{m}^{(1)}(y;\mathtt{t}) = \widetilde{m}_L^{(1)}(y;\mathtt{t})$ is given by

$$\widetilde{m}^{(1)}(y;\mathtt{t}) = \widetilde{m}^{(0)}(y;\mathtt{t}) \wedge \left(\bigwedge_{\omega^*(l\beta+\mathtt{t}) \in \{\boldsymbol{\Omega}^*\}^1(\mathtt{t}) \cap \overline{\Lambda}} m_{\omega^*(l\beta+\mathtt{t}),\mathtt{p}^1\times\mathtt{s}}(y)\right) \tag{3.17}$$

and

$$\{\boldsymbol{\Omega}^*\}^1(\mathtt{t}) = \left\{\omega^*(l\beta+\mathtt{t}) \in \{\boldsymbol{\Omega}^*\}(\mathtt{t}):\ \widetilde{t}^{(1)}(y;\mathtt{t}) = \mathtt{p}^1\right\} \tag{3.18}$$

yielding that

$$\{\boldsymbol{\Omega}^*\}^1(\mathtt{t}) \cap \overline{\Lambda} = \{P^1, \ldots, P^{\kappa_s}\}.$$

(Recall, the initial shift-vector is $s = (\mathtt{s}, 0)$ where $\mathtt{s} \in [0, 1/2)$.)

Pictorially, as before, the function $y \in \mathbb{R}^2 \mapsto \widetilde{m}^{(1)}(y;\mathtt{t})$ indicates by how much a particle at point y could be moved when we take into account the particles placed at points $\omega^*(l\beta+\mathtt{t}) \in \{\mathbf{\Omega}^*\}(\mathtt{t}) \cap \overline{\Lambda}^{\mathrm{c}}$ (which do not move) and the particles placed at points $\omega^*(l\beta+\mathtt{t}) \in \{\mathbf{\Omega}^*\}^1(\mathtt{t}) \cap \overline{\Lambda}$ (which are moved by $\mathtt{p}^1$). Consequently, $\widetilde{t}^{(2)}(y;\mathtt{t})$ indicates how much a movement by quantity $\widetilde{t}^{(0)}(y;\mathtt{t})$ should be reduced in presence of hard-core particles at points $\omega^*(l\beta+\mathtt{t}) \in \{\mathbf{\Omega}^*\}(\mathtt{t}) \cap \overline{\Lambda}^{\mathrm{c}}$ and $\omega^*(l\beta+\mathtt{t}) \in \{\mathbf{\Omega}^*\}^1(\mathtt{t}) \cap \overline{\Lambda}$.

Next, we minimise the function $\widetilde{t}^{(2)}(\,\cdot\,;\mathtt{t})$ over the $\mathtt{t}$-section $(\{\mathbf{\Omega}^*\}(\mathtt{t}) \setminus \{\mathbf{\Omega}\}^1(\mathtt{t})) \cap \overline{\Lambda}$ and, like before, set:

$$\begin{aligned} \mathtt{p}^2 &= \min\left[\widetilde{t}^{(2)}_L(y;\mathtt{t}):\ y \in (\{\mathbf{\Omega}^*\}(\mathtt{t}) \setminus \{\mathbf{\Omega}\}^1(\mathtt{t})) \cap \overline{\Lambda}\right], \\ P^{\kappa_1+1} &= \arg\min\left[\widetilde{t}^{(2)}(y;\mathtt{t}):\ y \in (\{\mathbf{\Omega}^*\}(\mathtt{t}) \setminus \{\mathbf{\Omega}\}^1(\mathtt{t})) \cap \overline{\Lambda}\right]. \end{aligned} \tag{3.19}$$

Again, if the minimum is shared by more than one point in $(\{\mathbf{\Omega}^*\}(\mathtt{t}) \setminus \{\mathbf{\Omega}\}^1(\mathtt{t})) \cap \overline{\Lambda}$, we list all these points: $P^{\kappa_1+1}, \ldots, P^{\kappa_1+\kappa_2}$ (in any order). As earlier, the value $\mathtt{p}^2$ is assigned to each of those points as $\mathtt{p}(P^j)$:

$$\begin{aligned} \mathtt{p}(\omega^*(l\beta+\mathtt{t})) = \mathtt{p}^2, \ \text{ if } \ &\omega^* \in \mathbf{\Omega}^*, 0 \leq l < k(\omega^*), \\ &\omega^*(l\beta+\mathtt{t}) \in \overline{\Lambda} \ \text{ and } \ \widetilde{t}^{(2)}(\omega^*(l\beta+\mathtt{t});\mathtt{t}) = \mathtt{p}^2. \end{aligned}$$

And so on: this procedure is iterated until we exhaust all points in $\{\mathbf{\Omega}^*\}(\mathtt{t}) \cap \overline{\Lambda}$. (Recall, their number and their positions vary with $\mathtt{t} \in [0,\beta]$.) At the end, we obtain a resulting function $\widetilde{t} = \widetilde{t}_L(\,\cdot\,;\mathtt{t})$:

$$y \in \mathbb{R}^2 \mapsto \widetilde{t}(y) \ \text{ where } \ \widetilde{t}(y) = \widetilde{t}^{(0)}(y) \wedge \widetilde{m}(y) \tag{3.20}$$

where

$$\widetilde{m}(y) = \widetilde{m}_L(y;\mathtt{t}) = \bigwedge_{\omega^*(l\beta+\mathtt{t}) \in \{\mathbf{\Omega}^*\}(\mathtt{t})} m_{\omega^*(l\beta+\mathtt{t}),\mathtt{p}(\omega^*(l\beta+\mathtt{t}))\times\mathtt{s}}(y). \tag{3.21}$$

Here we set:

$$\mathtt{p}(\omega^*(l\beta+\mathtt{t})) = 0 \ \text{ when } \ \omega^*(l\beta+\mathtt{t}) \in \overline{\Lambda}^{\mathrm{c}}.$$

Observe that

$$\widetilde{t}_L(y;\mathtt{t}) = 0 \text{ for } y \in \overline{\Lambda}^{\mathrm{c}} \ \text{ and } \ \widetilde{t}_L(y;\mathtt{t}) = 1 \text{ for } y \in \Lambda_{R(L)}. \tag{3.22}$$

The Jacobian $J^{\pm}_L(\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c})$ of the transform $\mathtt{T}^{\pm}_L$ turns out to be of the form:

$$J^{\pm}_L(\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c}) = \exp\left[\int_0^\beta \mathrm{d}\mathtt{t} \sum_{\omega^*\in\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c}} \times \sum_{0\leq l<k^*(\omega)} \ln\left(1 \pm \mathtt{s} \times \big(\partial^1 \widetilde{t}_L\,\big)(\omega^*(l\beta+\mathtt{t});\mathtt{t})\right)\right] \tag{3.23}$$

where $(\partial^1 \widetilde{t}_L\,)(y)$ stands for the partial derivative $\dfrac{\partial \widetilde{t}_L}{\partial \mathtt{y}^1}(y;\mathtt{t})$, $y = (\mathtt{y}^1, \mathtt{y}^2)$. (The fact that the functions $\widetilde{t}_L$ are non-differentiable on sets of positive co-dimension is not an obstacle here because of involvement of Wiener's integration.) The crucial quantity $\Big[J^+_L(\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c})J^-_L(\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c})\Big]^{1/2}$ in Eqn (2.11) becomes

$$\Big[J^+_L(\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c})J^-_L(\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c})\Big]^{1/2} = \exp\left(\int_0^\beta \mathrm{d}\mathtt{t} \times \sum_{\omega^*\in\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c}} \sum_{0\leq l<k^*(\omega)} \ln\left\{1 - \Big[\mathtt{s}^2\big(\partial^1 \widetilde{t}_L\big)(\omega^*(l\beta+\mathtt{t});\mathtt{t})\Big]^2\right\}\right). \tag{3.24}$$

We see that the quantity (3.24) is close to 1 when we are able to check that the sum

$$\sum_{\omega^*\in\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c}} \sum_{0\leq l<k^*(\omega)} \int_0^\beta \mathrm{d}\mathtt{t}\Big[\big(\partial^1 \widetilde{t}_L\big)(\omega^*(l\beta+\mathtt{t});\mathtt{t})\Big]^2 \tag{3.25}$$

is close to 0.

We conclude this section with a straightforward assertion justifying the definition (3.3) that introduces the intermediate square $\overline{\Lambda}$.

Lemma 3.1. *Consider the events*

$$\mathcal{L}^{(1)}_L = \{\mathbf{\Omega}^* \in \mathcal{W}^*_{\mathtt{r}}(\mathbb{R}^2) :\ \alpha_{\mathbb{R}^2\setminus\overline{\Lambda}}(\omega^*) = 1\ \ \forall\, \omega^* \in \mathbf{\Omega}^*\ \ \textit{with}\ \ x(\omega^*) \in \Lambda^{\mathrm{c}}\} \tag{3.25.1}$$

and

$$\mathcal{L}_L^{(2)}=\{\mathbf{\Omega}^*\in\mathcal{W}_{\mathtt{r}}^*(\mathbb{R}^2):\ \alpha_{\Lambda_{R(L)}}(\omega^*)=1\ \ \forall\,\omega^*\in\mathbf{\Omega}^*\ \ \textit{with}\ \ x(\omega^*)\in\Lambda_0\}\tag{3.25.2}$$

In other words, (a) *for* $\mathbf{\Omega}^*\in\mathcal{L}_L^{(1)}$, *every loop* ω^* *from* $\mathbf{\Omega}^*$ *which starts at a point* $x(\omega^*)$ *outside square* Λ *does not reach square* $\overline{\Lambda}$, *while* (b) *for* $\mathbf{\Omega}^*\in\mathcal{L}_L^{(2)}$, *every loop* ω_0^* *from* $\mathbf{\Omega}_{\Lambda_0}^*$ *(which starts in* Λ_0*) does not leave square* $\Lambda_{R(L)}$. *Then, under condition* (1.4),

$$\lim_{L\to\infty}\mu_L(\mathcal{L}_L^{(1)})=\lim_{L\to\infty}\mu_L(\mathcal{L}_L^{(2)})=1,\tag{3.26}$$

$\forall\ \mu\in\mathfrak{K}(z,\beta)$.

Proof of Lemma 3.1. Both relations are proved in a similar way, so we discuss in detail one of them, say $\lim\limits_{L\to\infty}\mu_L(\mathcal{L}_L^{(1)})=1$. At first we write

$$\begin{aligned}&\mu(\mathcal{W}_{\mathtt{r}}^*\setminus\mathcal{L}_L^{(1)})\\ &\quad=\mu(\exists\text{ at least one loop }\omega^*\text{ with }x(\omega^*)\in\Lambda^{\rm c}\text{ reaching }\overline{\Lambda})\\ &\quad\leq\int\mu({\rm d}\mathbf{\Omega}^*)\sum_{\omega^*\in\mathbf{\Omega}_{\Lambda^{\rm c}}^*}\mathbf{1}\Big(\omega^*(\mathtt{t})\in\overline{\Lambda}\text{ for some }\ \mathtt{t}\in[0,k(\omega^*)\beta]\Big).\end{aligned}$$

By virtue of the Campbell theorem, the last integral equals

$$\int{\rm d}\omega^*\rho(\omega^*)\mathbf{1}\Big(x(\omega^*)\in\Lambda^{\rm c}\text{ but }\omega^*(\mathtt{t})\in\overline{\Lambda}\text{ for some }\ \mathtt{t}\in[0,k(\omega^*)\beta]\Big).$$

By the Ruelle bound (cf. Eqns (2.3.18.**I**)–(2.3.20.**I**)) this does not exceed

$$\int_{\Lambda^{\rm c}}{\rm d}x\int_{\mathcal{W}^*(x)}\mathbb{P}_x({\rm d}\omega^*)\frac{\overline{\rho}^{k(\omega^*)}}{k(\omega^*)}\mathbf{1}\Big(\omega^*(\mathtt{t})\in\overline{\Lambda}\text{ for some }\ \mathtt{t}\in[0,k(\omega^*)\beta]\Big).$$

Next, we observe that the loop ω^* with the endpoint $x=(\mathtt{x}^1,\mathtt{x}^2)\in\Lambda^{\rm c}$ (i.e., with $\max\,|\mathtt{x}^j|_{\rm m}\geq L$) can reach $\overline{\Lambda}$ only if at least one of its one-dimensional components (i.e., a scalar Brownian bridge with the endpoint $\mathtt{x}^j$, $j=1$ or 2) deviates from its origin by at least $(|\mathtt{x}^j|-L)+L^{3/4}$. Therefore, the last displayed expression is upper-bounded by

$$\begin{aligned}&2\times2\sum_{k\geq1}\frac{\overline{\rho}^k}{k\sqrt{2\pi\beta k}}\int_{L^{3/4}}^\infty{\rm d}\mathtt{x}\exp\ \big[-4\mathtt{x}^2/(2k\beta)\big]\\ &\qquad\leq\sum_{k\geq1}\frac{4\overline{\rho}^k}{\sqrt{2\pi}\,k}\ \frac{\exp\big[-L^{3/2}/(2k\beta)\big]}{L^{3/4}/\sqrt{k\beta}+\sqrt{L^{3/2}/(k\beta)\ +4/\pi}}\,.\end{aligned}\tag{3.27}$$

Here we have used an estimate for the (scalar) Brownian bridge $\mathtt{B}(\mathtt{t})$ with endpoints $\mathtt{0}$ and $\mathtt{y} \geq 0$: $\forall\ \mathtt{A} > \mathtt{y}$

$$\overline{\mathtt{P}}^{\beta k}_{0,\mathtt{y}}\Big\{ \sup\, [\mathtt{B}(\mathtt{t}) :\ 0 \leq \mathtt{t} \leq \beta k] \geq \mathtt{A} \Big\} = \frac{1}{\sqrt{2\pi\beta k}} e^{-(2\mathtt{A}-y)^2/(2\beta k)} \tag{3.28.1}$$

plus well-known estimates for the tail of the normal distribution (Mills ratio bounds): $\forall\ \mathtt{A} \in (0,\infty)$,

$$\frac{e^{-\mathtt{A}^2/2}}{\mathtt{A}+\sqrt{\mathtt{A}^2+2}} \leq \int_{\mathtt{A}}^{\infty} e^{-\mathtt{t}^2/2} \mathrm{d}\mathtt{t} \leq \frac{\mathtt{e}^{-\mathtt{A}^2/2}}{\mathtt{A}+\sqrt{\mathtt{A}^2+4/\pi}}. \tag{3.28.2}$$

It is not hard to see that the RHS of (3.27) tends to 0 as $L \to \infty$. This completes the proof of Lemma 3.1.

In what follows we will assume that an LC $\mathbf{\Omega}^*$ lies in $\mathcal{L}_L$. Together with (3.22) this will imply that the loops $\omega^* \in \mathbf{\Omega}^*$ with $x(\omega^*) \in \Lambda^{\mathrm{c}}$ remains unaffected by transformations $\mathtt{T}^{\pm}(s)$.

4. Estimates for the Jacobians

To guarantee assertions (A) and (Ca) of Theorem 2.2 we need to secure that the good set $\mathcal{G}_L$ carries a large measure and contains only those LCs $\mathbf{\Omega}^* \in \mathcal{W}^*(\mathbb{R}^2)$ for which the expression $J^+_L(\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^{\mathrm{c}}}) J^-_L(\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^{\mathrm{c}}})$ can be appropriately controlled. To this end, consider a random variable $\Sigma^{\mathrm{J}}(\mathbf{\Omega}^*) = \Sigma^{\mathrm{J}}_L(\mathbf{\Omega}^*)$ given by the RHS of (3.24):

$$\begin{aligned}\Sigma^{\mathrm{J}}(\mathbf{\Omega}^*) : \mathbf{\Omega}^* \mapsto \int_0^\beta \mathrm{d}\mathtt{t} \sum_{x\in\{\mathbf{\Omega}^*\}(\mathtt{t})} \Big[\big(\partial^1 \widetilde{t}_L\big)(x;\mathtt{t})\Big]^2 \\ = \sum_{\omega^*\in\mathbf{\Omega}^*} \int_0^\beta \mathrm{d}\mathtt{t} \sum_{0\leq l<k^*(\omega)} \Big[\big(\partial^1 \widetilde{t}_L\big)(\omega^*(l\beta+\mathtt{t});\mathtt{t})\Big]^2.\end{aligned} \tag{4.1}$$

The formal definition of the set $\mathcal{G}_L$ will require that the quantity $\Sigma^{\mathrm{J}}(\mathbf{\Omega}^*)$ is small (more precisely that some majorants for $\Sigma^{\mathrm{J}}(\mathbf{\Omega}^*)$ are small); see below. Formally, the property that $J^+_L(\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^{\mathrm{c}}}) J^-_L(\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^{\mathrm{c}}})$ is close to 1 follows from

Lemma 4.1. *If $\epsilon > 0$ is chosen small enough then the mean-value of $\Sigma^{\rm J}(\mathbf{\Omega}^*)$ vanishes as $L \to \infty$:*

$$\lim_{L\to\infty} \int \mu({\rm d}\mathbf{\Omega}^*)\Sigma^{\rm J}(\mathbf{\Omega}^*) = 0. \tag{4.2}$$

Proof of Lemma 4.1. Let us start with technical definitions. Given $\mathtt{t} \in [0,\beta]$ and $x, x' \in \{\mathbf{\Omega}^*\}(t)$, we write:

$$\left.\begin{array}{rl} x \leftrightarrow x' & \text{whenever } a < |x - x'| < a + \epsilon \\ \text{and} & \\ x \overset{\{\mathbf{\Omega}^*\}(\mathtt{t})}{\longleftrightarrow} x'' & \text{when there exists a collection of particles} \\ & x_0, \ldots, x_m \in \mathbf{\Omega}^* \text{ such that point } x_0 \text{ coincides} \\ & \text{with } x, \text{ point } x_m \text{ with } x'' \text{ and } \forall\ i = 1, \ldots, m \\ & x_{i-1} \text{ and } x_i \text{ satisfy } a < |x_i - x_{i-1}| < a + \epsilon. \end{array}\right\} \tag{4.3}$$

Recall, the values $z, \beta > 0$ are such that the bound (1.4) is satisfied. Referring below to a small $\epsilon > 0$ we mean conditions like this:

$$\frac{1}{4}\pi[(a + 2\epsilon)^2 - a^2](1 \vee \beta)\left(1 \vee \sum_{k\geq 1} \overline{\rho}^k k \Big/ (2\pi\beta)\right) < 1. \tag{4.4}$$

To assess the integral in (4.2), observe that one possibility for value $\widetilde{t}_L(\omega^*(l\beta + \mathtt{t}))$ is $\widetilde{t}_L^{(0)}(\omega^*(l\beta + \mathtt{t}))$; the opposite case is where $\widetilde{t}_L(\omega^*(l\beta + \mathtt{t}))$ equals $\widetilde{m}_L(\omega^*(l\beta + \mathtt{t}))$. See Eqns (3.8), (3.20). In the former case we have to deal with the derivative

$$\partial^1 \widetilde{t}_L^{(0)}(\omega^*(l\beta + \mathtt{t}); \mathtt{t}) = Z_L(|\omega^*(l\beta + \mathtt{t})|_{\rm m})$$

where

$$Z_L(r) = \frac{[q\,(r - R(L) - \epsilon - a/2)]^2}{\left[Q(\overline{L} - R(L) - \epsilon - a/2)\right]^2}\, \mathbf{1}(0 \leq r \leq \overline{L}). \tag{4.5}$$

In the second case we obtain that

$$\widetilde{t}_L(\omega^*(l\beta + \mathtt{t})) = \widetilde{m}_L(\omega^*(l\beta + \mathtt{t})),$$

and we have to use the structure of function $\widetilde{m}_L(\omega^*(l\beta + \mathtt{t}))$ (related to multiple minimisation as defined in Eqn (3.21)) to assess its derivative. Cf. Sect 6.7 in [11].

All in all, to verify (4.2) it suffices to check that

$$\lim_{L\to\infty}\int\mu(\mathrm{d}\boldsymbol{\Omega}^*)[\Sigma^{(1)}(\boldsymbol{\Omega}^*)+\Sigma^{(2)}(\boldsymbol{\Omega}^*)]=0. \tag{4.6}$$

Here variable $\Sigma^{(1)}=\Sigma^{(1)}_L$ is given by

$$\begin{aligned}\Sigma^{(1)}(\boldsymbol{\Omega}^*)&=\int_0^\beta \mathrm{d}\mathtt{t}\sum_{x\in\{\boldsymbol{\Omega}^*\}(\mathtt{t})}\tau_L(|x|_{\mathrm{m}})\\&=\sum_{\omega^*\in\boldsymbol{\Omega}^*}\int_0^{k(\omega^*)\beta}\tau_L(|\omega^*(\mathtt{t})|_{\mathrm{m}})\mathrm{d}\mathtt{t}\end{aligned} \tag{4.7}$$

and corresponds to the first of the aforementioned possibilities (where we have $\widetilde{t}_L(\omega^*(l\beta+\mathtt{t}))=\widetilde{t}^{(0)}_L(\omega^*(l\beta+\mathtt{t})))$. The function τ_L Next, variable $\Sigma^{(2)}=\Sigma^{(2)}_L$ corresponds with the second possibility and has the form

$$\begin{aligned}\Sigma^{(2)}(\boldsymbol{\Omega}^*)=\int_0^\beta \mathrm{d}\mathtt{t}&\sum_{\substack{x,x',x''\in\{\boldsymbol{\Omega}^*\}(\mathtt{t})\\ x\neq x'}}\mathbf{1}\Big(x\leftrightarrow x'\Big)\mathbf{1}\Big(x\overset{\{\boldsymbol{\Omega}^*\}(\mathtt{t})}{\longleftrightarrow}x''\Big)\\&\times\mathbf{1}\Big(|x|_{\mathrm{m}}\leq|x''|_{\mathrm{m}}\Big)\Big[\tau_L(|x|_{\mathrm{m}}-\epsilon-a/2)-\tau_L(|x''|_{\mathrm{m}})\Big]^2\\&:=\Sigma^{(2,1)}(\boldsymbol{\Omega}^*)+\Sigma^{(2,2)}(\boldsymbol{\Omega}^*).\end{aligned} \tag{4.8}$$

The composition of the RHS is related to a 'cluster' structure accompanying the multiple minimization procedure in (3.21) which determines the value of interest $\widetilde{m}_L(\omega^*(l\beta+\mathtt{t}))$. Formally, as follows from the definition, behind the indicator $\mathbf{1}\Big(x\overset{\{\boldsymbol{\Omega}^*\}(\mathtt{t})}{\longleftrightarrow}x''\Big)$ there is a 'chain' of points from the $\mathtt{t}$-section $\{\boldsymbol{\Omega}^*\}(\mathtt{t})$ which joins the 'extreme' points x and x''. Cf. Sect. 8 in [11] (whose system notation is partially followed here).

Moreover, the partition $\Sigma^{(2)}(\boldsymbol{\Omega}^*)=\Sigma^{(2,1)}(\boldsymbol{\Omega}^*)+\Sigma^{(2,2)}(\boldsymbol{\Omega}^*)$ reflects the fact that x and x'', the two extreme points in the chain, can belong to the same loop ω^* or to two distinct loops, ω^* and $\omega^{*\prime\prime}$. More precisely, the summand

$\Sigma^{(2,1)} = \Sigma_L^{(2,1)}$ is specified as the sum

$$\begin{aligned}
&\sum_{\omega^*\in\mathbf{\Omega}^*}\int_0^\beta \mathrm{d}\mathtt{t} \sum_{0\le l,l''<k(\omega^*)} \mathbf{1}\left(\omega^*(l\beta+\mathtt{t})\overset{\{\mathbf{\Omega}^*\}(\mathtt{t})}{\longleftrightarrow}\omega^*(l''\beta+\mathtt{t})\right)\\
&\quad\times\Bigg[\mathbf{1}(\omega^*(l\beta+\mathtt{t})\leftrightarrow\omega^*(l''\beta+\mathtt{t}))\mathbf{1}(l\neq l'')\\
&\qquad+\sum_{\omega^{*\prime}\neq\omega^{*\prime\prime}}\sum_{0\le l'<k(\omega^{*\prime})}\mathbf{1}(\omega^*(l\beta+\mathtt{t})\leftrightarrow\omega^{*\prime}(l'\beta+\mathtt{t}))\Bigg]\\
&\qquad\times\mathbf{1}\Big(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}}\le|\omega^*(l''\beta+\mathtt{t})|_{\mathrm{m}}\Big)\\
&\quad\times\Big[\tau_L(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}}-\epsilon-a/2)-\tau_L(|\omega^*(l''\beta+\mathtt{t})|_{\mathrm{m}})\Big]^2
\end{aligned} \tag{4.9.1}$$

whereas the term $\Sigma^{(2,2)} = \Sigma_L^{(2,2)}$ equals the sum

$$\begin{aligned}
&\sum_{\omega^*,\omega^{*\prime\prime}\in\mathbf{\Omega}^*}\int_0^\beta \mathrm{d}\mathtt{t} \sum_{\substack{0\le l<k(\omega^*)\\0\le l''<k(\omega^{*\prime\prime})}} \mathbf{1}\left(\omega^*(l\beta+\mathtt{t})\overset{\{\mathbf{\Omega}^*\}(\mathtt{t})}{\longleftrightarrow}\omega^{*\prime\prime}(l''\beta+\mathtt{t})\right)\\
&\times\Bigg[\mathbf{1}(\omega^*(l\beta+\mathtt{t})\leftrightarrow\omega^{*\prime\prime}(l''\beta+\mathtt{t}))\\
&+\sum_{0\le l'<k(\omega^{*\prime\prime})}\mathbf{1}(l'\neq l'')\mathbf{1}(\omega^*(l\beta+\mathtt{t})\leftrightarrow\omega^{*\prime\prime}(l'\beta+\mathtt{t}))\\
&\quad+\sum_{\omega^{*\prime}\neq\omega^{*\prime\prime}}\sum_{0\le l'<k(\omega^{*\prime})}\mathbf{1}(\omega^*(l\beta+\mathtt{t})\leftrightarrow\omega^{*\prime}(l'\beta+\mathtt{t}))\Bigg]\\
&\quad\times\mathbf{1}\Big(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}}\le|\omega^{*\prime\prime}(l''\beta+\mathtt{t})|_{\mathrm{m}}\Big)\\
&\times\Big[\tau_L(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}}-\epsilon-a/2)-\tau_L(|\omega^{*\prime\prime}(l''\beta+\mathtt{t})|_{\mathrm{m}})\Big]^2.
\end{aligned} \tag{4.9.2}$$

Constants $C_j\in(0,\infty)$ appearing in the argument below vary with β and z (through $\overline{\rho}$) but are independent of L.

Proposition 4.1. *The mean value of $\Sigma^{(1)}$ is assessed as follows:*

$$\int_{\mathcal{W}_{\mathtt{r}}^*(\mathbb{R}^2)}\mu(\mathrm{d}\mathbf{\Omega}^*)\Sigma_L^{(1)}(\mathbf{\Omega}^*)\le C_0\gamma(L) \tag{4.9}$$

where $C_0\in(0,\infty)$ is a constant and the quantity $\gamma(L)$ is defined as follows:

$$\gamma(L):=\int_\Lambda\frac{q(|x|_{\mathrm{m}}-R(L)-\epsilon-a/2)^2}{Q(L-R(L)-\epsilon-a/2)^2}\mathrm{d}x,\quad\text{with}\quad\lim_{L\to\infty}\gamma(L)=0. \tag{4.10}$$

Proof of Proposition 4.1. To explain the bound (4.10), we first write, by the Campbell theorem:

$$\begin{aligned}&\int_{\mathcal{W}^*_{\mathrm{r}}(\mathbb{R}^2)}\mu(\mathrm{d}\mathbf{\Omega}^*)\Sigma_L^{(1)}(\mathbf{\Omega}^*)\\ &\qquad=\int_{\mathcal{W}^*}\mathrm{d}\omega^*\rho(\omega^*)\int_0^{k(\omega^*)\beta}\tau_L(|\omega^*(\mathtt{t})|_{\mathrm{m}})\mathrm{d}\mathtt{t}.\end{aligned}\tag{4.11}$$

By the Ruelle bound, the RHS does not exceed

$$\int_{\mathcal{W}^*}\mathrm{d}\omega^*\frac{\overline{\rho}^{k(\omega^*)}}{k(\omega^*)}\int_0^{k(\omega^*)\beta}\tau_L(|\omega^*(\mathtt{t})|_{\mathrm{m}})\mathrm{d}\mathtt{t}.\tag{4.12}$$

When $x(\omega^*)\in\Lambda_{R(L)}$, we estimate

$$Z_L(|\omega^*(\mathtt{t})|_{\mathrm{m}})\leq\frac{1}{\left[Q(L-R(L)-\epsilon-a/2)\right]^2};\tag{4.13}$$

consequently, the corresponding contribution

$$\int_{\mathcal{W}^*}\mathrm{d}\omega^*\mathbf{1}(x(\omega^*)\in\Lambda_{R(L)})\frac{\overline{\rho}^{k(\omega^*)}}{k(\omega^*)}\int_0^{k(\omega^*)\beta}\tau_L(|\omega^*(\mathtt{t})|_{\mathrm{m}})\mathrm{d}\mathtt{t}$$

does not exceed

$$\begin{aligned}&\frac{(2R(L))^2}{\left[Q(L-R(L)-\epsilon-a/2)\right]^2}\sum_{k\geq 1}\frac{(k\beta)\overline{\rho}^k}{(2\pi k\beta)k}\\ &\qquad=\frac{\big(\log\log\ L\big)^{3/2}}{2\pi\left[Q(L-R(L)-\epsilon-a/2)\right]^2}\sum_{k\geq 1}\frac{\overline{\rho}^k}{k}\\ &\qquad\qquad<\frac{\overline{\rho}\big(\log\log\ L\big)^{3/2}}{2\pi(1-\overline{\rho})\left[Q(L-R(L)-\epsilon-a/2)\right]^2}.\end{aligned}\tag{4.14}$$

This idea can be pushed further: we use estimate (4.14) whenever loop ω^* reaches $\Lambda_{R(L)}$. For given $x\notin\Lambda_{R(L)}$ and $\omega^*\in\mathcal{W}^*(x)$ this can occur when either (i) $k(\omega^*)$ is large – say, $k(\omega^*)>\big[|x|_{\mathrm{m}}-R(L)\big]/2$ – or when (ii) the opposite inequality $k(\omega^*)\leq\big[|x|_{\mathrm{m}}-R(L)\big]/2$ holds true but the loop ω^* deviates from x, in the max-distance, by at least $|x|_{\mathrm{m}}-R(L)$. Then the corresponding part of expression (4.14)

$$\begin{aligned}&\int_{\mathcal{W}^*}\mathrm{d}\omega^*\mathbf{1}(x(\omega^*)\notin\Lambda_{R(L)})\\ &\qquad\times\mathbf{1}(\omega^*(\mathtt{t})\in\Lambda_{R(L)}\ \text{ for some }\ t\in[0,k(\omega^*)\beta])\\ &\qquad\qquad\times\frac{\overline{\rho}^{k(\omega^*)}}{k(\omega^*)}\int_0^{k(\omega^*)\beta}\tau_L(|\omega^*(\mathtt{t})|_{\mathrm{m}})\mathrm{d}\mathtt{t}\end{aligned}$$

is upper-bounded by

$$\int_{\mathbb{R}^2} \mathrm{d}x \Big\{ \sum_{k\geq |x|_{\mathrm{m}}/2} \frac{(k\beta)\overline{\rho}^k}{(2\pi k\beta)k} + \sum_{1\leq k\leq |x|_{\mathrm{m}}/2} \frac{(k\beta)\overline{\rho}^k}{k} \int_{\mathcal{W}^{k\beta}(0)} \mathbb{P}_0^{k\beta}(\mathrm{d}\omega^*) \\ \times \mathbf{1}\Big(\max\,\big[|\omega^*(\mathtt{t})|_{\mathrm{m}}:\ 0\leq \mathtt{t}\leq k\beta\big] > |x|_{\mathrm{m}}\Big)\Big\}. \tag{4.15}$$

The first sum in (4.15) is evaluated through a convergent geometric progression:

$$\sum_{k\geq |x|_{\mathrm{m}}/2} \frac{\overline{\rho}^k}{2\pi k} \leq \frac{\overline{\rho}^{|x|_{\mathrm{m}}/2}}{2\pi(1-\overline{\rho})},$$

and its contribution into the integral $\displaystyle\int_{\mathbb{R}^2}\mathrm{d}x$ does not exceed a constant. To estimate the second sum, one can use the inequalities (3.27.1,2). This yields:

$$\sum_{1\leq k\leq |x|_{\mathrm{m}}^{1/2}} \frac{(k\beta)\overline{\rho}^k}{k} \int_{\mathcal{W}^{k\beta}(0)} \mathbb{P}_0^{k\beta}(\mathrm{d}\omega^*) \\ \times \mathbf{1}\Big(\max\,\big[|\omega^*(\mathtt{t})|_{\mathrm{m}}:\ 0\leq \mathtt{t}\leq k\beta\big] > |x|_{\mathrm{m}}\Big) \\ \leq \frac{2}{|x|_{\mathrm{m}}+\sqrt{|x|_{\mathrm{m}}^2+4/\pi}}\,\frac{e^{-|x|_{\mathrm{m}}/\beta}}{2\pi(1-\overline{\rho})}. \tag{4.16}$$

Consequently, the contribution of this sum to $\displaystyle\int_{\mathbb{R}^2}\mathrm{d}x$ also does not exceed a constant.

More generally, for a given $r > R(L)$ we consider the contribution into (4.14) from loops ω^* with $x(\omega^*)\not\in\Lambda_r$ such that $|\omega^*(\mathtt{t})|_{\mathrm{m}} = r$ for some $\mathtt{t}\in[0,k(\omega^*)\beta]$. Repeating the above argument, we conclude that this contribution again is less than or equal to a constant times $\tau_L(r)$. Note that all constants can be made uniform; this implies that

$$(4.14) \leq \frac{C_0}{[Q(L-R(L)-\epsilon-a/2)]^2} \\ \times\Big[\big(\log\log\ L\big)^{3/2} + \int_{R(L)}^{L} [q\,(r-R(L)-\epsilon-a/2)]^2\,\mathrm{d}r\Big]. \tag{4.17}$$

As in [11], the quantity in the RHS of (4.17) (which is $= C_0\gamma(L)$) goes to 0 as $L\to\infty$. This finishes the proof of Proposition 4.1.

It is instructive to note that the relation (4.9) does not require a smallness for ϵ.

We now pass to random variable $\Sigma_L^{(2)} = \Sigma_L^{(2,1)} + \Sigma_L^{(2,2)}$.

Proposition 4.2. *For ϵ small enough,*

$$\lim_{L\to\infty} \int_{\mathcal{W}^*_{\mathtt{r}}(\mathbb{R}^2)} \mu(\mathrm{d}\mathbf{\Omega}^*)\Sigma_L^{(2)}(\mathbf{\Omega}^*) = 0. \tag{4.18}$$

Proof of Proposition 4.2. In the beginning, we again use the Campbell theorem (in conjunction with an argument similar to Eqn (6.25) from [11]). Then the integral in (4.18) is less than or equal to a constant (say, C_1) times the sum $I^{2,1} + I^{2,2}$. Here the term $I^{2,1} = I_L^{2,1}$ is specified as follows:

$$\begin{aligned}
I^{2,1} = &\int_0^\beta \mathrm{d}\mathtt{t} \int \mathrm{d}\omega^* \sum_{0\le l,l''<k(\omega^*)} \Bigg\{ \mathbf{1}(\omega^*(l\beta+\mathtt{t}) \leftrightarrow \omega_1^*(l''\beta+\mathtt{t}))\rho(\omega^*) \\
&+ \sum_{m\ge 1} \prod_{1<i\le m} \int \mathrm{d}\omega_i^* \sum_{0\le l_i,\bar{l}_i<k(\omega_i^*)} \mathbf{1}(\omega_{i-1}^*(\bar{l}_{i-1}\beta+\mathtt{t}) \leftrightarrow \omega_i^*(l_i\beta+\mathtt{t})) \\
&\quad \times \mathbf{1}(\omega_m^*(l_m\beta+\mathtt{t}) \leftrightarrow \omega^*(l''\beta+\mathtt{t}))\rho(\omega^*,\omega_1^*,\ldots,\omega_m^*) \Bigg\} \\
&\qquad \times \mathbf{1}\Big(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}} \le |\omega^*(l''\beta+\mathtt{t})|_{\mathrm{m}} \Big) \\
&\quad \times \Big[\tau_L(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}} - \epsilon - a/2) - \tau_L(|\omega^*(l''\beta+\mathtt{t})|_{\mathrm{m}}) \Big]^2
\end{aligned} \tag{4.19.1}$$

where the loop ω_0^* has been identified as ω^* and value $\bar{l}_0$ as l.

Likewise, the summand $I^{2,2} = I^{2,2}_L$ is given by

$$\begin{aligned}
I^{2,2} = &\int_0^\beta \mathrm{d}\mathtt{t} \int \mathrm{d}\omega^* \int \mathrm{d}\omega^{*\prime\prime} \\
&\times \sum_{\substack{0\le l<k(\omega^*)\\ 0\le l''<k(\omega^{*\prime\prime})}} \Big\{ \mathbf{1}(\omega^*(l\beta+\mathtt{t}) \leftrightarrow \omega^{*\prime\prime}(l''\beta+\mathtt{t}))\rho(\omega^*,\omega^{*\prime\prime}) \\
&+\sum_{m\ge1}\prod_{1<i\le m}\int \mathrm{d}\omega_i^* \sum_{0\le l_i,\bar{l}_i<k(\omega_i^*)} \mathbf{1}(\omega_{i-1}^*(\bar{l}_{i-1}\beta+\mathtt{t}) \leftrightarrow \omega_i^*(l_i\beta+\mathtt{t})) \\
&\times \mathbf{1}(\omega_m^*(l_m\beta+\mathtt{t}) \leftrightarrow \omega^{*\prime\prime}(l''\beta+\mathtt{t}))\rho(\omega^*,\omega_1^*,\dots,\omega_m^*,\omega^{*\prime\prime}) \Big\} \\
&\quad \times \mathbf{1}\Big(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}} \le |\omega^{*\prime\prime}(l''\beta+\mathtt{t})|_{\mathrm{m}}\Big) \\
&\times \Big[\tau_L(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}} - \epsilon - a/2) - \tau_L(|\omega^{*\prime\prime}(l''\beta+\mathtt{t})|_{\mathrm{m}})\Big]^2
\end{aligned} \tag{4.19.2}$$

where again the loop ω_0^* has been identified as ω^* and value $\bar{l}_0$ as l.

So, it suffices to verify that

$$\lim_{L\to\infty} I^{2,1} = \lim_{L\to\infty} I^{2,2} = 0.$$

Both integrals are analysed in a similar fashion, and we focus on one of them, say, $I^{2,2}$.

We use elementary bounds

$$\begin{aligned}
&\Big[\tau_L(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}} - \epsilon - a/2) - \tau_L(|\omega^{*\prime\prime}_{\ m}(l''\beta+\mathtt{t})|_{\mathrm{m}})\Big]^2 \\
&\quad \le \Big[|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}} - \epsilon - a/2 - |\omega^{*\prime\prime}(l''\beta+\mathtt{t}))|_{\mathrm{m}}\Big]^2 Z_L(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}})
\end{aligned} \tag{4.20.1}$$

with Z_L given in (4.5), and

$$\begin{aligned}
&\Big[|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}} - \epsilon - a/2 - |\omega^{*\prime\prime}(l''\beta+\mathtt{t}))|_{\mathrm{m}}\Big]^2 \le 3(\epsilon+a/2)^2 \\
&\quad +3|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}}^2 + 3|\omega^{*\prime\prime}(l''\beta+\mathtt{t}))|_{\mathrm{m}}^2.
\end{aligned} \tag{4.20.2}$$

Employing in addition the Ruelle bound, we conclude that (4.19.2) does

not exceed

$$
\begin{aligned}
&3\int_0^\beta \mathrm{d}\mathtt{t}\int \mathrm{d}\omega^*\frac{\overline{\rho}^{k(\omega^*)}}{k(\omega^*)}\int \mathrm{d}\omega^{*\prime\prime}\frac{\overline{\rho}^{k(\omega^{*\prime\prime})}}{k(\omega^{*\prime\prime})}\\
&\quad\times\sum_{\substack{0\le l<k(\omega^*)\\0\le l''<k(\omega^{*\prime\prime})}} Z_L(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}})\Big\{\mathbf{1}(\omega^*(l\beta+\mathtt{t})\leftrightarrow\omega^{*\prime\prime}(l''\beta+\mathtt{t}))\\
&\qquad+\sum_{m\ge1}\prod_{1<i\le m}\int \mathrm{d}\omega_i^*\frac{\overline{\rho}^{k(\omega_i^*)}}{k(\omega_i^*)}\\
&\qquad\times\sum_{0\le l_i,\bar{l}_i<k(\omega_i^*)}\mathbf{1}(\omega_{i-1}^*(\bar{l}_{i-1}\beta+\mathtt{t})\leftrightarrow\omega_i^*(l_i\beta+\mathtt{t}))\\
&\quad\times\mathbf{1}(\omega_m^*(l_m\beta+\mathtt{t})\leftrightarrow\omega^{*\prime\prime}(l''\beta+\mathtt{t}))\Big\}\\
&\qquad\times\mathbf{1}\Big(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}}\le|\omega^{*\prime\prime}(l''\beta+\mathtt{t})|_{\mathrm{m}}\Big)\\
&\quad\times\Big[(\epsilon+a/2)^2+|\omega_0^*(l_0\beta+\mathtt{t})|_{\mathrm{m}}^2+|\omega_m^*(l_m\beta+\mathtt{t}))|_{\mathrm{m}}^2\Big]\,.
\end{aligned}
\tag{4.21}
$$

Expanding the sum of squares in the parentheses, we obtain three expressions; in view of similarity of the argument used for analysing each of them, we focus on the one with the term $|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}}^2$:

$$
\begin{aligned}
&\int_0^\beta \mathrm{d}\mathtt{t}\int \mathrm{d}\omega^*\frac{\overline{\rho}^{k(\omega^*)}}{k(\omega^*)}Z_L(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}})\ \int \mathrm{d}\omega^{*\prime\prime}\frac{\overline{\rho}^{k(\omega^{*\prime\prime})}}{k(\omega^{*\prime\prime})}\\
&\qquad\times\mathbf{1}\Big(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}}\le|\omega^{*\prime\prime}(l''\beta+\mathtt{t})|_{\mathrm{m}}\Big)\\
&\quad\times\sum_{\substack{0\le l<k(\omega^*)\\0\le l''<k(\omega^{*\prime\prime})}}|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}}^2\Big\{\mathbf{1}(\omega^*(l\beta+\mathtt{t})\leftrightarrow\omega^{*\prime\prime}(l''\beta+\mathtt{t}))\\
&\qquad+\sum_{m\ge1}\prod_{1<i\le m}\int \mathrm{d}\omega_i^*\frac{\overline{\rho}^{k(\omega_i^*)}}{k(\omega_i^*)}\\
&\qquad\times\sum_{0\le l_i,\bar{l}_i<k(\omega_i^*)}\mathbf{1}(\omega_{i-1}^*(\bar{l}_{i-1}\beta+\mathtt{t})\leftrightarrow\omega_i^*(l_i\beta+\mathtt{t}))\\
&\quad\times\mathbf{1}(\omega_m^*(l_m\beta+\mathtt{t})\leftrightarrow\omega^{*\prime\prime}(l''\beta+\mathtt{t}))\Big\}.
\end{aligned}
\tag{4.22}
$$

Again, we can expand the curled brackets and will analyse the behavior of

the technically most involved sum that emerges:

$$\begin{aligned}&\int_0^\beta \mathrm{d}\mathtt{t}\int \mathrm{d}\omega^*\frac{\overline{\rho}^{k(\omega^*)}}{k(\omega^*)}\int \mathrm{d}\omega^{*\prime\prime}\frac{\overline{\rho}^{k(\omega^{*\prime\prime})}}{k(\omega^{*\prime\prime})}\\ &\times\sum_{\substack{0\leq l<k(\omega^*)\\ 0\leq l''<k(\omega^{*\prime\prime})}}|\omega^*(l\beta+\mathtt{t})|^2_{\mathrm{m}}Z_L(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}})\\ &\quad\times\sum_{m\geq 1}\prod_{1<i\leq m}\int \mathrm{d}\omega_i^*\frac{\overline{\rho}^{k(\omega_i^*)}}{k(\omega_i^*)}\\ &\quad\times\sum_{0\leq l_i,\bar{l}_i<k(\omega_i^*)}\mathbf{1}(\omega^*_{i-1}(\bar{l}_{i-1}\beta+\mathtt{t})\leftrightarrow\omega_i^*(l_i\beta+\mathtt{t}))\\ &\times\mathbf{1}(\omega^*_m(l_m\beta+\mathtt{t})\leftrightarrow\omega^{*\prime\prime}(l''\beta+\mathtt{t})).\end{aligned}\tag{4.23}$$

The argument for estimating (4.23) starts with the analysis of the integral $\displaystyle\int \mathrm{d}\omega^{*\prime\prime}\frac{\overline{\rho}^{k(\omega^{*\prime\prime})}}{k(\omega^{*\prime\prime})}$ for fixed values of the variables in the remaining integrals. To this end, we invoke the Fubini theorem and properties of the Brownian bridge. This allows us to conclude that

$$\begin{aligned}&\int \mathrm{d}\omega^{*\prime\prime}\frac{\overline{\rho}^{k(\omega^{*\prime\prime})}}{k(\omega^{*\prime\prime})}\sum_{0\leq l''<k(\omega^{*\prime\prime})}\mathbf{1}(\omega^*_m(l_m\beta+\mathtt{t})\leftrightarrow\omega^{*\prime\prime}(l''\beta+\mathtt{t}))\leq\\ &\int_{\mathbb{R}^2}\mathrm{d}x''\int_0^\beta \mathrm{d}\mathtt{t}\int_{A[\omega_{m-1}(\mathtt{t}),\epsilon]}\mathrm{d}y\sum_{k''\geq 1}\overline{\rho}^{k''}\frac{e^{-|y-x''|^2/(2\mathtt{t})}}{2\pi}\frac{e^{-|y-x''|^2/(2(k''\beta-\mathtt{t}))}}{2\pi(k''\beta-\mathtt{t})}.\end{aligned}\tag{4.24}$$

Here

$$A[\omega_{m-1}(\mathtt{t}),\epsilon]=\{y\in\mathbb{R}^2:a<|y-\omega_{m-1}(\mathtt{t})|<a+2\epsilon\}\tag{4.25}$$

stands for an annulus of width 2ϵ around the center $\omega_{m-1}(\mathtt{t})$. (Initially, point $y\in A[\omega_{m-1}(\mathtt{t}),\epsilon]$ emerges here as the point on the circle of radius $|y-\omega_{m-1}(\mathtt{t})|$ about $\omega_{m-1}(\mathtt{t})$ where the loop ω hits this circle while t is the hitting time.) The RHS of (4.24) yields a quantity $\leq C_2\epsilon$.

This argument can be iterated for the integrals $\displaystyle\int \mathrm{d}\omega_i^*\frac{\overline{\rho}^{k(\omega_i^{*\prime\prime})}}{k(\omega_i^{*\prime\prime})}$ where we have to take into account the double sum $\displaystyle\sum_{0\leq l_i,\bar{l}_i<k(\omega_i^*)}$. However, it only affects the constant in front of ϵ.

At the end, assuming that $\epsilon > 0$ is small enough we arrive at the following bound for (4.23):

$$\frac{C_3\epsilon}{1-C_4\epsilon}\int_0^\beta \mathrm{d}\mathtt{t}\int \mathrm{d}\omega^* \frac{\overline{\rho}^{k(\omega^*)}}{k(\omega^*)}\sum_{0\leq l<k(\omega^*)} |\omega^*(l\beta+\mathtt{t})|^2_{\mathrm{m}} Z_L(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}}). \quad (4.26)$$

The integral (4.26) is analysed in the same manner as in Proposition 4.1 (cf. (4.9)) and tends to 0. (The presence of the sum $\sum\limits_{0\leq l<k(\omega^*)}$ in (4.26) does not affect the core of the argument.)

This completes the proof of Proposition 4.2 and Lemma 4.1.

5. Estimates for the change in the energy. Concluding remarks

In this section we assess the expression

$$\exp\ \Big[h(\mathtt{T}^+_L(s)\Omega^*_\Lambda|\boldsymbol{\Omega}^*_{\Lambda^c}) + h(\mathtt{T}^-_L(s)\Omega^*_\Lambda|\boldsymbol{\Omega}^*_{\Lambda^c}) - 2h(\Omega^*_\Lambda|\boldsymbol{\Omega}^*_{\Lambda^c})\Big]$$

(cf. (2.8)). The argument is based on the same idea as in Sect 8.6 of [11] (again we partially borrow the system of notation from there). In the course of the argument we will produce a further (and final) specification of the set $\mathcal{G}_L \subset \mathcal{W}^*_{\mathtt{r}}(\mathbb{R}^2)$ of good LCs. Namely, given $\boldsymbol{\Omega}^* \in \mathcal{W}^*_{\mathtt{r}}(\mathbb{R}^2)$, we set, as before,

$$\boldsymbol{\Omega}^*_\Lambda = \{\omega^* \in \boldsymbol{\Omega}^* :\ x(\boldsymbol{\omega}^*) \in \Lambda\}, \quad \boldsymbol{\Omega}^*_{\Lambda^c} = \{\omega^* \in \boldsymbol{\Omega}^* :\ x(\boldsymbol{\omega}^*) \in \Lambda^c\}.$$

Then write

$$\begin{aligned} &h(\mathtt{T}^+_L(s)\boldsymbol{\Omega}^*_\Lambda|\boldsymbol{\Omega}^*_{\Lambda^c}) + h(\mathtt{T}^-_L(s)\boldsymbol{\Omega}^*_\Lambda|\boldsymbol{\Omega}^*_{\Lambda^c}) - 2h(\boldsymbol{\Omega}^*_\Lambda|\boldsymbol{\Omega}^*_{\Lambda^c}) \\ &\quad = \int_0^\beta \mathrm{d}\mathtt{t}\Big\{E[\{\mathtt{T}^+(s)\boldsymbol{\Omega}^*_\Lambda\}(\mathtt{t})|\{\boldsymbol{\Omega}^*_{\Lambda^c}\}(\mathtt{t})] \\ &\qquad +E[\{\mathtt{T}^-(s)\boldsymbol{\Omega}^*_\Lambda\}(\mathtt{t})|\{\boldsymbol{\Omega}^*_{\Lambda^c}\}(\mathtt{t})] - 2E[\{\boldsymbol{\Omega}^*_\Lambda\}(\mathtt{t})|\{\boldsymbol{\Omega}^*_{\Lambda^c}\}(\mathtt{t})]\Big\}. \end{aligned} \quad (5.1)$$

Here $E\left[\{\mathtt{T}^\pm(s)\boldsymbol{\Omega}^*\}_\Lambda(\mathtt{t})|\{\boldsymbol{\Omega}^*_{\Lambda^c}\}(\mathtt{t})\right]$ is defined as the sum

$$\begin{aligned} &\frac{1}{2}\sum_{x,x'\in\{\boldsymbol{\Omega}^*_\Lambda\}(\mathtt{t})} V\left(|x \pm s\widetilde{t}(x) - x' \mp s\widetilde{t}(x')|\right) \\ &\qquad + \sum_{\substack{x\in\{\boldsymbol{\Omega}^*_\Lambda\}(\mathtt{t})\\ x'\in\{\boldsymbol{\Omega}^*_{\Lambda^c}\}(\mathtt{t})}} V\left(|x \pm s\widetilde{t}(x) - x' \mp s\widetilde{t}(x')|\right) \end{aligned} \quad (5.2)$$

while $E[\{\mathbf{\Omega}^*\}(\mathtt{t})|\mathbf{\Omega}^*_{\Lambda^c}(\mathtt{t})]$ is obtained by omitting the terms containing the shift-vector s. Cf. (2.3.11.**I**)–(2.3.12.**I**). Recall, our aim is to guarantee that on the good set $\mathcal{G}_L$, the absolute values of the variables $\Sigma_L^{(i)}(\mathbf{\Omega}^*)$ are small. Two straightforward bounds turn out to be helpful:

$$\begin{array}{l}\Big|E[\{\mathtt{T}^+(s)\mathbf{\Omega}^*_\Lambda\}(\mathtt{t})|\{\mathbf{\Omega}^*_{\Lambda^c}\}(\mathtt{t})] + E[\{\mathtt{T}^-(s)\mathbf{\Omega}^*_\Lambda\}(\mathtt{t})|\{\mathbf{\Omega}^*_{\Lambda^c}\}(\mathtt{t})] \\ \qquad\qquad -2E[\{\mathbf{\Omega}^*_\Lambda\}(\mathtt{t})|\{\mathbf{\Omega}^*_{\Lambda^c}\}(\mathtt{t})]\Big| \leq \overline{V}^{(2)}\mathtt{s}^2 \\ \quad\times\left\{\frac{1}{2}\sum\limits_{x,x'\in\{\mathbf{\Omega}^*_\Lambda\}(\mathtt{t})} + \sum\limits_{\substack{x\in\{\mathbf{\Omega}^*_\Lambda\}(\mathtt{t})\\ x'\in\{\mathbf{\Omega}^*_{\Lambda^c}\}(\mathtt{t})}}\right\}|\widetilde{t}(x)-\widetilde{t}(x')|^2\mathbf{1}(|x-x'|\leq \mathrm{R}_0)\end{array} \tag{5.3}$$

and

$$\begin{array}{l}\frac{1}{3}|\widetilde{t}(x)-\widetilde{t}(x')|^2 \\ \quad\leq |\widetilde{t}(x)-\tau_L(|x|_{\mathrm{m}})|^2 + |\tau_L(|x|_{\mathrm{m}})-\tau_L(|x'|_{\mathrm{m}})|^2 + |\tau_L(|x'|_{\mathrm{m}})-\widetilde{t}(x')|^2.\end{array} \tag{5.4}$$

These bounds yield that

$$\begin{array}{l}\big|h(\mathtt{T}^+_L(s)\mathbf{\Omega}^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^c}) + h(\mathtt{T}^-_L(s)\mathbf{\Omega}^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^c}) - 2h(\mathbf{\Omega}^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^c})\big| \\ \leq 3\overline{V}^{(2)}\mathtt{s}^2\displaystyle\int_0^\beta \mathrm{d}\mathtt{t}\left\{\frac{1}{2}\sum_{x,x'\in\{\mathbf{\Omega}^*_\Lambda\}(\mathtt{t})} + \sum_{\substack{x\in\{\mathbf{\Omega}^*_\Lambda\}(\mathtt{t})\\ x'\in\{\mathbf{\Omega}^*_{\Lambda^c}\}(\mathtt{t})}}\right\}\mathbf{1}(|x-x'|\leq \mathrm{R}_0) \\ \qquad\times\Big[2|\widetilde{t}(x)-\tau_L(|x|_{\mathrm{m}})|^2 + |\tau_L(|x|_{\mathrm{m}})-\tau_L(|x'|_{\mathrm{m}})|^2\Big] \\ \qquad\qquad =: \Sigma_L^{(3)}(\mathbf{\Omega}^*) + \Sigma_L^{(4)}(\mathbf{\Omega}^*)\end{array} \tag{5.5}$$

where variables $\Sigma_L^{(3)}$ and $\Sigma_L^{(4)}$ emerge when we expand the sum of squares in the parentheses.

As above, we will try to make sure that the expected values of variables $\Sigma_L^{(3)}$ and $\Sigma_L^{(4)}$ vanish as $L\to\infty$:

Lemma 5.1.

$$\lim_{L\to\infty}\int\mu(\mathrm{d}\mathbf{\Omega}^*)\Sigma_L^{(3)}(\mathbf{\Omega}^*) = \lim_{L\to\infty}\int\mu(\mathrm{d}\mathbf{\Omega}^*)\Sigma_L^{(4)}(\mathbf{\Omega}^*) = 0. \tag{5.6}$$

Proof of Lemma 5.1. As before, we focus on one of the relations in Eqn (5.6), say, for $\Sigma_L^{(4)}$. It is instructive to expand

$$\Sigma_L^{(4)}(\mathbf{\Omega}^*) = \Sigma_L^{(4,1)}(\mathbf{\Omega}^*) + \Sigma_L^{(4,2)}(\mathbf{\Omega}^*).$$

Here $\Sigma_L^{(4,1)}(\mathbf{\Omega}^*)$ gives a single-loop contribution to $\Sigma_L^{(4)}(\mathbf{\Omega}^*)$ while $\Sigma_L^{(4,2)}(\mathbf{\Omega}^*)$ yields a contribution from pairs of loops:

$$\begin{aligned}\Sigma_L^{(4,1)}(\mathbf{\Omega}^*) = &\sum_{\omega^*\in\mathbf{\Omega}^*_\Lambda}\int_0^\beta \mathrm{d}\mathtt{t}\Bigg\{\sum_{0\le l<\bar{l}<k(\omega^*)}\\ &\times\Big[\tau_L\big(\big|\omega^*(l\beta+\mathtt{t})\big|_{\mathrm{m}}\big)-\tau_L\big(\big|\omega^*(\bar{l}\beta+\mathtt{t})\big|_{\mathrm{m}}\big)\Big]^2\\ &\quad\times\mathbf{1}\Big(\big|\omega^*(l\beta+\mathtt{t})-|\omega^*(\bar{l}\beta+\mathtt{t}\big|<\mathrm{R}_0\Big)\\ &+\sum_{\omega^{*\prime}\in\mathbf{\Omega}^*_{\Lambda^{\rm c}}}\sum_{\substack{0\le l<k(\omega^*)\\ 0\le l'<k(\omega^{*\prime})}}\Big[\tau_L\big(\big|\omega^*(l\beta+\mathtt{t})\big|_{\mathrm{m}}\big)-\tau_L\big(\big|\omega^{*\prime}(l'\beta+\mathtt{t})\big|_{\mathrm{m}}\big)\Big]^2\\ &\quad\times\mathbf{1}\Big(\big|\omega^*(l\beta+\mathtt{t})-|\omega^{*\prime}(l'\beta+\mathtt{t}\big|<\mathrm{R}_0\Big)\Bigg\}\end{aligned}\tag{5.7.1}$$

and

$$\begin{aligned}\Sigma_L^{(4,2)}(\mathbf{\Omega}^*) = \frac{1}{2}&\int_0^\beta \mathrm{d}\mathtt{t}\sum_{\substack{\omega^*,\omega^{*\prime}\in\mathbf{\Omega}^*_\Lambda\\ \omega^*\ne\omega^{*\prime}}}\sum_{\substack{0\le l<k(\omega^*)\\ 0\le l'<k(\omega^{*\prime})}}\\ &\times\Big[\tau_L\big(\big|\omega^*(l\beta+\mathtt{t})\big|_{\mathrm{m}}\big)-\tau_L\big(\big|\omega^{*\prime}(l'\beta+\mathtt{t})\big|_{\mathrm{m}}\big)\Big]^2\\ &\quad\times\mathbf{1}\Big(\big|\omega^*(l\beta+\mathtt{t})-|\omega^{*\prime}(l'\beta+\mathtt{t}\big|<\mathrm{R}_0\Big).\end{aligned}\tag{5.7.2}$$

(The factor $3\overline{V}^{(2)}\mathtt{s}^2$ carried from (5.5) has been discarded.)

Following Eqn (6.22) from [11], we estimate: (a) when $\big|\omega^*(l\beta+\mathtt{t})\big|_{\mathrm{m}}\le\big|\omega^{*\prime}(l'\beta+\mathtt{t})\big|_{\mathrm{m}}$,

$$\begin{aligned}&\Big[\tau_L\big(\big|\omega^*(l\beta+\mathtt{t})\big|_{\mathrm{m}}\big)-\tau_L\big(\big|\omega^{*\prime}(l'\beta+\mathtt{t})\big|_{\mathrm{m}}\big)\Big]^2\\ &\le\Big[\big|\omega^*(l\beta+\mathtt{t})\big|_{\mathrm{m}}-\big|\omega^{*\prime}(l'\beta+\mathtt{t})\big|_{\mathrm{m}}-\epsilon-a/2\Big]^2 Z_L(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}})\end{aligned}$$

and (b) when $\big|\omega^{*\prime}(l'\beta+\mathtt{t})\big|_{\mathrm{m}}\le\big|\omega^*(l\beta+\mathtt{t})\big|_{\mathrm{m}}$,

$$\begin{aligned}&\Big[\tau_L\big(\big|\omega^*(l\beta+\mathtt{t})\big|_{\mathrm{m}}\big)-\tau_L\big(\big|\omega^{*\prime}(l'\beta+\mathtt{t})\big|_{\mathrm{m}}\big)\Big]^2\\ &\le\Big[\big|\omega^{*\prime}(l'\beta+\mathtt{t})\big|_{\mathrm{m}}-\big|\omega^*(l\beta+\mathtt{t})\big|_{\mathrm{m}}-\epsilon-a/2\Big]^2 Z_L(|\omega^{*\prime}(l\beta+\mathtt{t})|_{\mathrm{m}})\end{aligned}$$

where Z_L has been defined in (4.5).

After substituting these estimates in (5.1), the relation $\displaystyle\int \mu(\mathrm{d}\mathbf{\Omega}^*)\Sigma_L^{(4)}(\mathbf{\Omega}^*)$ $\to 0$ is verified in the same way as in Proposition 4.1. This completes the proof of Lemma 5.1.

Lemma 5.1 (and the comments on other terms emerging from the bound (5.5)), together with Lemmas 3.1 and 4.1, allows us to define the set $\mathcal{G}_L$. Namely,

$$\mathcal{G}_L = \Big\{\mathbf{\Omega}^* \in \mathcal{L}_L :\ \Sigma_L^{(i)}(\mathbf{\Omega}^*) < c,\ 1 \leq i \leq 4\Big\} \tag{5.8}$$

where $c \in (0, \infty)$ is a chosen constant (viz., $c = 1/2$). Applying the Chebyshev inequality guarantees

Lemma 5.2. *$\forall\ \delta \in (0,1)$ and $c \in (0,\infty)$, there exists $L_1^* \in (0,\infty)$ such that for $L > L_1^*$ the probability $\mu(\mathcal{G}_L) \geq 1 - \delta$.*

A formal summary of properties of transformations $\mathtt{T}^{\pm}(s)$ is given in Theorem 5.1:

Theorem 5.1. *Given $\mathbf{\Omega}^* \in \mathcal{G}_L$, the transformations $\mathtt{T}_L^{\pm}(s) :\ \mathbf{\Omega}^* \mapsto \widetilde{\mathbf{\Omega}}^*$ $\in \mathcal{W}_{\mathtt{r}}^*(\mathbb{R}^2)$ possess the following properties:*

(i) *The maps $\mathtt{T}^{\pm}(s)$ are measurable and $1-1$.*

(ii) *$\widetilde{\mathbf{\Omega}}^*_{\Lambda^{\mathrm{c}}} = \mathbf{\Omega}^*_{\Lambda^{\mathrm{c}}}$ and $\widetilde{\mathbf{\Omega}}^*_{\Lambda_0} = \mathtt{S}(s)\mathbf{\Omega}^*_{\Lambda_0}$. Moreover, there exists a $1-1$ correspondence between the loops $\widetilde{\omega}^* \in \widetilde{\mathbf{\Omega}}^*_{\Lambda}$ and $\omega^* \in \mathbf{\Omega}^*_{\Lambda}$ such that $\widetilde{\omega}^*$ is obtained as a deformation of ω^* via tuned shifts of $\mathtt{t}$-sections, in the manner described in Section 3. In particular, $k(\widetilde{\omega}^*) = k(\omega^*)$.*

(iii) *The equality (2.6) holds true, where the expression*
$\Big[J_L^+(\Omega^*_{\Lambda} \vee \mathbf{\Omega}^*_{\Lambda^{\mathrm{c}}})J_L^-(\Omega^*_{\Lambda} \vee \mathbf{\Omega}^*_{\Lambda^{\mathrm{c}}})\Big]^{1/2}$ *is close to 1 uniformly in $\mathbf{\Omega}^*$ for L large.*

(iv) *The quantity (2.8) is close to 1 uniformly in $\mathbf{\Omega}^*$ when L is large enough.*

The assertion of Theorem 2.1 then follows.

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